Many-Sorted Implicative Conceptual Systems

JAN ODELSTAD

Doctoral Thesis
Stockholm, Sweden 2008
Abstract

A theory of many-sorted implicative conceptual systems (abbreviated msic-systems) is presented. Examples of msic-systems include legal systems, normative systems, systems of rules and instructions, and systems expressing policies and various kinds of scientific theories. In computer science, msic-systems can be used in, for instance, legal information systems, decision support systems, and multi-agent systems. In the thesis, msic-systems are studied from a logical and algebraic perspective aiming at clarifying their structure and developing effective methods for representing them. Of special interest are the most narrow links or joinings between different strata in a system, that is between subsystems of different sorts of concepts, and the intermediate concepts intervening between such strata. Special emphasis is put on normative systems, and the role that intermediate concepts play in such systems, with an eye on knowledge representation issues. Normative concepts are constructed out of descriptive concepts using operators based on the Kanger-Lindahl theory of normative positions. An abstract architecture for a norm-regulated multi-agent system is suggested, containing a scheme for how normative positions will restrict the set of actions that the agents are permitted to choose from. Technical results include a characterization of an msic-system in terms of the most narrow joinings between different strata, characterization of the structure of the most narrow joinings between two strata, conditions for the extendability of intermediate concepts, and finally, a specification of the conditions such that the Boolean operations on intermediate concepts will result in intermediate concepts and characterization of most narrow joinings in terms of weakest grounds and strongest consequences.
Contents

Introduction vii

1 msic-systems
Outline and Comments 1

1 msic-systems - an overview 3
  1.1 Preamble ............................................. 3
  1.2 The theory of msic-systems .................... 4
  1.3 Normative systems ............................. 5
    1.3.1 What is a norm? ............................ 5
    1.3.2 Norms in predicate logic and as ordered pairs .... 6
  1.4 Conceptual systems ............................ 9
  1.5 Intermediate concepts-form and function ...... 10
    1.5.1 Intermediaries ............................. 10
    1.5.2 Open intermediaries ..................... 14
    1.5.3 Intermediaries in normative systems ......... 16
    1.5.4 Implicative closeness between strata ......... 16
  1.6 Deontic consequences .......................... 18
    1.6.1 Deontic logic with the action operator Do ... 18
    1.6.2 Normative positions ........................ 19
    1.6.3 Normative systems as msic-systems ........... 21
  1.7 The algebraic approach to msic-systems ...... 22
    1.7.1 Set-theoretical predicates ................. 22
    1.7.2 Boolean quasi-orderings and joining systems .. 22
    1.7.3 Models and variations of the algebraic theories .. 24
  1.8 Applications in computer science ............. 25
    1.8.1 Introduction .................................. 25
    1.8.2 Normative systems and computer science .... 25
    1.8.3 Agent oeconomicus norma .................... 27
    1.8.4 Normative positions regulating actions ....... 28
    1.8.5 Prolog implementation of norm-regulated DLMAS ... 30
  1.9 Norms and forest cleaning .................... 31
    1.9.1 Introduction ................................ 31
1.9.2 Cleaning in silico ........................................ 31

2 Comments on the papers ........................................ 35
  2.1 The formal representation of msic-systems ............... 35
  2.2 Related work .............................................. 36
    2.2.1 Intermediate concepts .................................. 36
    2.2.2 The representation of normative systems ............... 37
    2.2.3 Norms and artificial agent-systems .................... 38
  2.3 Non-Boolean joining systems ................................ 38
    2.3.1 Joining systems of equality-relations ................. 38
    2.3.2 Joining systems of aspects ............................ 39
  2.4 Two comments on DALMAS .................................. 40
    2.4.1 The move-operator ...................................... 40
    2.4.2 A note on Prohibited ................................... 40

II The Papers .................................................... 45

A Proofs of the results in paper (5) ............................ 49
  A.0.3 Weakest grounds and strongest consequences ........... 49
  A.0.4 Minimality .............................................. 50
  A.0.5 Interventions ............................................ 52

References ....................................................... 57
Acknowledgements

There would never have been a theory of msic-systems without Lars Lindahl’s collaboration and there would never have been any Dalmas-architecture or dissertation without Magnus Boman’s support and assistance. Many thanks to both of you!

As my dissertation supervisor, Magnus helped me take the step from philosophy to computer science and made me realize that even artificial agents need norms. Advice from and discussions with Magnus have always been very rewarding. Magnus has also helped me with the practical problems that arise from completing a dissertation from a far, not least in the printing of the dissertation.

My collaboration with Lars stretches over decades and has spanned across different issues and taking various forms, including the same research projects. The exchange of thoughts with Lars has always been extremely inspiring and has influenced me in innumerable ways. In the Introduction, I describe, in greater detail, how this collaboration resulted in the co-authored essays in this dissertation.

Heartfelt thanks to Ulla Ahonen-Jonmarth and Magnus Hjelmblom for cooperative efforts in implementing the practice of the theories contained in this dissertation and for inspiring discussions about norm governance in decision making processes.

I also would like to thank colleagues at the Department of Mathematics, Natural and Computer Sciences at the University of Gävle for an inspiring intellectual setting. A special thanks to Eva Carling for how she simplified my move from philosopher to computer scientist. Similar thanks go to Anita Hussénius and Stefan Seipel for how they created the institutional circumstances to complete this dissertation.

For financial support, I would like to thank Gävle University, the KK-foundation and the Swedish Research Council.

Last but not least, a huge thanks to Jakob and Elias for your merry indulgence and to Lena for your tolerance this time too.

I dedicate this book with love to my mother.

Gävle, October 2008

Jan Odelstad
Introduction

Goals and methodology

In the famous Schilpp-volume where established scholars discuss Einstein’s work in physics and philosophy, Einstein, in his reply to criticisms, states the following about the relationship between epistemology and science:

The reciprocal relationship of epistemology and science is of noteworthy kind. They are dependent upon each other. Epistemology without contact with science becomes an empty scheme. Science without epistemology is—insofar it is thinkable at all—primitive and muddled. However, no sooner has the epistemologist, who is seeking a clear system, fought his way through to such a system, than he is inclined to interpret the thought-content of science in the sense of his system and to reject whatever does not fit into his system. The scientist, however, cannot afford to carry his striving for epistemological systematic that far. He accepts gratefully the epistemological conceptual analysis; but the external conditions, which are set for him by the facts of experience, do not permit him to let himself be too much restricted in the construction of his conceptual world by the adherence to an epistemological system. He therefore must appear to the systematic epistemologist as a type of unscrupulous opportunist ... (Einstein, 1949.)

The science Einstein has in mind is primarily physics, but even for sciences that are rather unlike physics its reciprocal relationship to epistemology is of a noteworthy kind. The external conditions that, according to Einstein, restrict the adherence to an epistemological system is “the facts of experience”, with what Einstein probably meant the results of observations and experiments. But for the sciences that are rather unlike physics “the facts of experience” may better be characterized in some other way. For computer- and systems sciences, which is the subject of this thesis, “the facts of experience” may perhaps be described as “useful applications”.

Every science ought to critically question its foundational assumptions. How urgent the researchers in a field experience these foundational questions may vary greatly from time to time. But probably all sciences go through stages when
the need for revisions and elaborations of the basic principles and fundamental conceptions seem inevitable. In a young science, the foundational problems are important and at the same time not seldom overlooked, since researchers working in the field are so enthusiastic over the flow of new results. In such situations, philosophy (which includes epistemology as one of its sub-disciplines) may have a role to play to make clear—and sometimes even to remedy—weak points in the base of the new discipline. This thesis has as its goal to address some problems in the foundations of computer- and systems sciences and construct theories and tools which could be useful in the further development of some aspects of this discipline.

Concepts are a fundamental tool for all kinds of human communication and concept formation is an important process in all branches of science. Information science is of course not an exception. An information system is, when all technical “embeddings” have been stripped off, a set of concepts and relations between these concepts. The skeleton of an information system is a conceptual structure, and this structure must have a solid formal representation, otherwise it cannot function in a computer context.

One of the themes in this thesis is the formal representations of conceptual systems. This theme has a long history in philosophy and in several scientific disciplines. In this thesis, the contribution to the study of conceptual systems is focused on the relation between layers or strata of concepts of different sorts in a conceptual system and on intermediate concepts that function as links between different strata. This study is brought about using algebraic tools, which implies that the representation is algebraic in character. The result is a theory of many-sorted implicational conceptual systems, mfsis-systems.

I argue for an anti-nivelistic approach to theoretical systems, which implies the recognition of the multitude of layers or strata that usually are parts of such systems. As a consequence, I also argue for an anti-nivelistic approach to knowledge representation. The following sketch is very vague and metaphorical, however, my message is more adequately found in the formalism of the thesis. Suppose that an mfsis-system M represents knowledge or information of a domain D. The implicative relation between concepts represents knowledge of some kind and the kind of knowledge it represents may differ in different parts of the system. In some parts of the system, it may represent conceptual knowledge, the knowledge of definitions of concepts and the logical relations between concepts. In other parts of the system, it may represent for example empirical knowledge about some kind of phenomena and in yet another part of the system it may represent empirical knowledge of another kind. Different strata of concepts of different sorts may thus express knowledge of different kinds. The knowledge represented by links between different strata often represent knowledge of a kind still different from the knowledge represented by the strata, for example knowledge of rational actions or appropriate rules. The revision of an mfsis-system can be done very partially. In many cases, the necessary revision

---

1 Nivelistic is constructed out of the French verb nivelier, meaning “Mettre au même niveau, rendre égal”.
is effected by the modification of the narrowest links between some strata of different kinds.

It is often argued that, for example, rule-based expert systems cannot be modified by the expert system itself. The following quotation from a textbook may illustrate this idea:

Knowledge in a rule-based expert system is represented by IF-THEN production rules collected by observing or interviewing human experts. This task, called knowledge acquisition, is difficult and expensive. In addition, once the rules are stored in the knowledge base, they cannot be modified by the expert system itself. Expert systems cannot learn from experience or adapt to new environments. Only a human can manually modify the knowledge base by adding, changing or deleting some rules. (Negnevitsky, 2005, p. 261.)

One of the advantages with the anti-nivelistic approach to knowledge representation expressed by msic-systems is, as I see it, that this may not be true. This is discussed in connection with forest cleaning in section 1.10.

The papers included in the thesis

The papers included in the thesis are the following:


In the co-authored papers together with Lars Lindahl that are included in this thesis, the following passage is incorporated:

The paper, as well as our earlier joint papers, are the result of wholly joint work; the order of appearance of our author names has no significance.
Short descriptions of the papers

The first co-authored publication on what is here called m sic-system (the term is not used in our publications) is from 1996. Since then we have dealt with the subject several times and published a number of papers, wherein we have developed our ideas about the subject further. In our earlier publications we focused on intermediate concepts and the work on the formal representation of m sic-systems was mainly to create a framework for our study of intermediaries. But we found that the formal representation of m sic-systems was an interesting subject in its own right and we devoted some later papers to this subject (especially to the representation of normative systems), among others the first and third papers in this thesis. The second paper in the thesis, which is an abridged and improved version of my master thesis in computer- and systems sciences and written in collaboration with Magnus Boman, applies the theory developed in paper (1) as a part of an abstract architecture for norm-regulated multiagent-systems. Paper (3) discusses revisions of normative systems mainly in the form of comments on a legal example. In paper (4), the structural m sic-theory provides the framework for a study of intermediate concepts, especially the negation of intermediaries and the subclass of open intermediaries. In paper (5), we take a first step towards a general theory of the stratification of m sic-systems with special emphasis on the role of intermediate concepts.

Notes on the conception of this thesis

I will give a short account of the background of this thesis, which can be appropriate since this is my second thesis for the doctor’s degree (the first was in theoretical philosophy at the University of Uppsala).

Besides basic concepts and theories in universal algebra, the Scandinavian discussion about intermediate concepts within the philosophy of law (see section 5 of chapter 1 and paper (4) section 1.2) has been the main inspiration of this thesis. I still remember my first encounter with this discussion. It was in the student café Ubbbo in Uppsala in the late 70s. At that time Lars Lindahl and I were participating in a research project about justice under the direction of Stig Kanger. I was working on the notion of the weight of factors in an aggregation of different aspects, a problem within multi-criteria analysis. It struck me that the notion of ‘weight’ applied to factors was being used in two ways, one normative for stating the weight a factor ought to have in an aggregation, and one descriptive for describing the weight a factor has in a certain aggregation, but that the distinction between the two uses of ‘weight’ was often obscured. However, I did not think that this was all that should be said about different uses of the notion of ‘weight’ in this context, and I tried to apply the traditional philosophical distinction between Realgrund and Erkenntnisgrund to this

---

2I found Ingemar Hedénius’ distinction between genuinely normative and spuriously normative sentences illuminating in this context. This distinction was introduced in Hedénius (1941).
problem. When I told Lars about my thoughts regarding the weight of factors, he advised me to, instead of applying that distinction, consider the use of the ideas of Ekelöf, Ross, Wedberg and others on intermediate concepts in the philosophy of law. These ideas were not so easy to grasp immediately, but as time passed I was more and more convinced that the theory of intermediate concept is important not only in jurisprudence but in science in general.

In the 80s, I was very influenced by P.W. Bridgman’s operationalistic approach to concept formation and it seemed to me that operationalism and the ideas about intermediate concepts fit well together in roughly the following manner: If a predicative concept is neither purely normative nor operationally definable, consider if it is an intermediate concept. I tried this dictum on various concepts and now and then I found the results interesting. Especially the notion of ‘probability’ was fascinating in this respect. Sentences like ‘the probability of the event e is r’ is on the one hand used descriptively, reporting on someone’s estimation, and on the other hand prescriptively, stating a kind of prescription or disposition of how to act in certain situations. ‘Probability’ used in the second sense seemed to me to be an intermediate concept linking observations of facts (descriptive grounds) about the event e (for example the relative frequency with which it has occurred in the future) to obligations to act in certain ways (normative consequences) given certain conditions. In the literature there has been extensive discussion on the interpretation of ‘probability’, and it struck me that the so-called objective or frequency interpretation is a theory of the grounds of probability-statements while the so-called subjective or personalistic interpretation is a theory of the consequences of probability-statements. I have developed this line of thought in a unpublished manuscript, Odelstad (1989) and this idea is mentioned in Lindahl & Odelstad (1999a), section 4.

During my “experiments” with intermediate concepts, the discussions with Lars and the reading of his papers on the subject (Lindahl 1968 and 1985) were an inexhaustible source of inspiration for me. When I wrote, in the 80s and 90s, my book on the weighing of interests (Odelstad, 2002), Lars and I had extensive discussions on the problems I was dealing with and thus also about intermediate concepts. One important theme in this book is how society tries to govern urban planning using norms. In such norms, intermediate concepts like ‘private interest’ and ‘public interest’ play an essential role. In the book, I also tried a more general application of the idea of intermediate concepts to decision theory: I studied aggregations of aspects and some aspects turned out to be intermediate concepts defined in terms of component-relations to ground aspects and to consequence aspects (see Odelstad 2002, p. 364).

In the middle of the 90s, Lars and I started to develop a formal theory of intermediate concepts. Using lattice theory and later the theory of Boolean algebras we introduced a theory of how intermediate concepts could function as more or less narrow links between different conceptual structures (see Lindahl & Odelstad 1996, 1999a, 2000, and Odelstad & Lindahl 1998 and 2000). Our

3Percy Williams Bridgman, 1882-1961, American physicist and philosopher of science, received the Nobel Prize in physics 1946.
main applications were taken from legal science and we considered intermediate concepts primarily as constituents in normative systems. As a result of this, we began to focus more on the formal representation of normative systems than on intermediate concepts, and the book by Carlos Alchourrón and Eugenio Bulygin (Alchourrón & Bulygin 1971) was an important source of inspiration for us. The work reported on in the paper (1) was commenced in the autumn of 2001 and in this paper there is no discussion of intermediate concepts.

With the exception of the academic year 2001-2002, when Lars held a research fellowship at the Swedish Collegium for Advanced Study in the Social Sciences (situated in Uppsala), much of the basic work on our different co-authored papers was done during a number of stays at Sigtunastiftelsen, often lasting one week or longer. Sigtunastiftelsen is a marvelous place for intellectual work with, among other things, a rose-garden and two rooms in the tower well-suited for bold conjectures and second thoughts, and we usually worked all day and a substantial part of the night. In the meantime, between the stays in Sigtuna we discussed our joint work often in very long telephone calls, innumerable emails and preliminary drafts.

Our lattice-based theory of intermediate concepts and the formal representations of normative systems were presented at the Fourth International Conference on Deontic Logic in Computer Science, DEON'98 in Bologna. I then realised that much of the most interesting research on the formal aspects of norms was performed in the borderland between computer science, mathematics and philosophy, and I found the applications of such formal theories in different parts of computer- and systems sciences very interesting. When the KK-foundation through the program ‘The promotion of research in IT’ in the late 90s offered academic teachers who already had a doctor’s exam the possibility to “switch competence” by taking a master’s degree in computer science or related disciplines and to start doing research in this field, I successfully applied for such a grant. I asked Magnus Boman, whom I knew had done research about norms for artificial agents, to supervise my master’s degree (and later also doctor’s degree) and fortunately Magnus agreed. Magnus introduced me to the research on artificial multi-agent systems and the norm-regulation of such systems, and in my master’s thesis I tried to apply results from Lars’s and my research about normative systems to norm-regulations of artificial multi-agent systems. Paper (2) in this thesis, which is co-authored with Magnus Boman, is an improved version of my master’s thesis.

The architecture called DALMAS of a norm-regulated multi-agent system presented in my master’s thesis awoke the interest of one of my colleagues, Magnus Hjelmblom, at the Division of Computer Science at the University of Gävle. Magnus Hjelmblom has implemented DALMAS in Prolog and the result is an executable logic program. Through the work of Magnus Hjelmblom, Magnus Boman and I have been fortunate to see DALMAS come alive. Consequently, a lot of interesting experiments with normative systems and their revisions are now within reach. See further section 1.8.5 and chapter 2 for a discussion of different aspects of the implementation of DALMAS.

Theory and practice—as well as the interplay between them—have always
interested me. Concrete applications of the theoretical work about norms and artificial agents presented in this thesis is, from my point of view, a fascinating subject and through the cooperation with one of my colleagues, Ulla Ahonen-Jonnarth at the Division of Computer Science at the University of Gävle (Ulla is a “competence switcher” like myself), a somewhat unexpected application is developing: the automation of the decision making executed by human or artificial forest cleaners. “Norm-like systems” and intermediate concepts seem to be useful for the construction of effective decision principles that can be automatically improved by machine learning. (See section 1.9.)

During the last four years, my teaching duties at the University of Gävle have mainly been the administration, development and teaching of the master program Decision, risk and policy analysis. This has given me the opportunity to return to my earlier interest in decision theory and its applications and I think, which is obvious already from Odelstad (2002), that intermediate concepts have a role to play in this context. To take a decision is, from my point of view and somewhat metaphorically expressed, to perform a walk in a network of open intermediaries. (Open intermediate concepts are discussed in paper 4.)

In my doctor’s thesis in theoretical philosophy (Odelstad, 1992), I studied different kinds of dependence relations between aspects (attributes) represented as systems of relationalss. One part of my book on the weighing of interests (Odelstad 2002) is devoted to the aggregation of aspects. One of my research plans for the future is to try to combine the results from these two books with the results in this thesis about msic-systems. The hypothesis is that in many contexts (for example in multi-criteria analysis) we are dealing with systems of aspects of different sorts and some of the aspects are intermediaries joining strata of aspects of different sorts. (A short remark about this is found in chapter 2.) If an aspect is an intermediate concept the same may hold for the numerical representations of this aspect. Hence, measures or scales for aspects can be intermediate concepts, for example joining descriptive grounds to normative consequences. This can be one approach for analysing the so called summary measures, which is the subject of an ongoing research project I am pursuing together with Johan Bring and Stig Bjomskog.

Another one of my research plans for the future is the application of msic-systems in decision and agent theory, aiming at a theory of “decision analytic expert systems using norms” with applications in forest cleaning, a project planned in collaboration with Ulla Ahonen-Jonnarth.
Part I

\textit{msic-systems}

Outline and Comments
Chapter 1

\textit{msic-systems - an overview}

1.1 Preamble

In an article from 1936, Albert Einstein discusses, among other things, the stratification of the scientific system. According to Einstein, there is a multitude of different layers or strata of concepts in science, where higher layers are more abstract than lower layers. As regards to the final aim of science, Einstein suggests, intermediary layers are only of temporal nature and must eventually disappear as irrelevant. But in the science of today, these strata represent partial success, though problematic. (See Einstein, 1973, p. 295.)

Many theoretical systems show the same kind of phenomena as theoretical physics in the following respects: In the system there is a hierarchical ordering of the concepts in different strata and the status of the concepts in intermediate strata is not obvious. In theoretical physics, the ordering of the layers is based on degrees of abstraction. In other contexts, the stratification of the system can be grounded on quite different principles, for example: descriptive versus normative, state versus action or physical versus mental. One of the main issues to be examined in this thesis is the stratification of concepts in theoretical systems, especially the connections between different strata and the function and status of intermediate layers.

The kind of theoretical systems that will come into focus in this study can, in a fairly general way, be characterized as \textit{conceptual systems} and two essential characteristics of these systems are the following: They have an implicative form and they are many-sorted, i.e. a system consists of different sorts of concepts (at least two). They are thus \textit{many-sorted implicative conceptual systems}, in the sequel abbreviated \textit{msic-systems}, that this thesis is about.\footnote{The term ‘many-sorted implicative conceptual system’ (taken together as one unit) and its abbreviation ‘msic-system’ are here introduced for the first time, they are neither found in the papers (1)-(5) nor in any other publication I have been involved in.} Different kinds of systems belong to the class under study, for example legal systems, normative systems, systems of rules and instructions, systems expressing policies and some
varieties of scientific theories. Such systems have an important role to play in the discipline artificial intelligence, which has as one of its aims to bring forth “smart” behaviour of computers.

In the investigation reported here, msic-systems are studied from a logical and algebraic perspective aiming at clarifying their structure and developing effective methods for representing them. Special emphasis is put on the most “narrow” links between subsystems of different sorts in a system and intermediaries (intermediate concepts) mediating or intervening between subsystems of different sorts. Such links and intermediaries are of great interest when there are reasons for changing the system.

In computer science, msic-systems can be useful in many problem areas, for example: legal information systems, computer security, knowledge representation, expert systems, architectures for multiagent-systems, decision-analytic support systems and agent-based simulations. This study of msic-systems is mainly a contribution to the tradition of constructing intelligible and explicit models and representations in contrast to case-based, connectivist and emergent approaches (cf. Luger, p. 228). But msic-systems also prepare the grounds for the use of machine learning, where the links and intermediaries between subsystems will play an important role.

The work that is presented in this thesis could be described from two different perspectives. It can start from the abstract core and proceed to the more and more concrete models and applications. Or it can start from the concrete problems that constitute the raison d’être from a practical point of view. I will here switch back and forth between these two perspectives.

1.2 The theory of msic-systems

A theory of msic-systems is developed in this thesis. Different parts of the theory are situated on different levels of abstraction, and as a consequence there are different levels of applications of the theory. When developing a theory of msic-systems, it is important to remember that the word ‘theory’ has several meanings and in this context it is important to distinguish between the following two meanings:

(1) Theory in the sense often used in logic; abstract theory, theory in contrast to model (in the model-theoretic sense)

(2) Theory in contrast to practice and application.

In this thesis, a theory of msic-systems is put forward in both senses of ‘theory’. The theory of msic-systems, where ‘theory’ is taken in the second sense contains some theories of msic-systems in the first sense, of formal theories. The formal theories of msic-systems are characterized axiomatically as algebraic theories and among the models of these abstract (formal) theories are specific msic-systems. The abstract theories of msic-systems express the structure of such
1.3. NORMATIVE SYSTEMS

systems. The theory of msic-systems, in the second sense, contains other theo-
retical perspectives than the abstract, formal ones.

The theory of msic-systems will be abbreviated msic-theory, where ‘theory’
stands for sense (2). A formal, abstract theory of msic-systems where ‘theory’
is taken in sense (1) will be called a structural msic-theory, since such a theory
characterizes the structure of msic-systems. Such abstract theories will usually
be presented as axiomatized theories within set theory. The most abstract part
of the msic-theory will be framed as a number of set-theoretical predicates, see
section 1.7.

1.3 Normative systems

1.3.1 What is a norm?

The theory of msic-systems has many applications and there are many different
kinds of msic-systems. The papers in this thesis focus on the representations
of normative systems as msic-systems. In this section, we take a first step in
the analysis of norms and a great deal of simplification is needed. Modifications
and elaborations of this oversimplified picture will be developed step by step in
later sections of the thesis.

Norms, normative sentences, are understood in contrast to descriptive sen-
tences. Sentences of the latter kind express matters of fact but are not used for
expressing evaluations or value judgments. A normative sentence, on the other
hand, does not state what is the case but what shall be the case or what may
be the case, or will have an evaluating function.

Let us preliminarily say that there are two kinds of normative sentences,
viz. categorically normative sentences and conditional normative sentences. A
categorically normative sentence consists of a descriptive sentence preceded by
a ‘norm creating operator’, for example ‘it shall be the case that’ or ‘it may be
the case that’. If q is a descriptive sentence then ‘it shall be the case that q’,
which is abbreviated Shall(q) and ‘it may be the case that q’, abbreviated as
May(q), are examples of categorically normative sentences. A conditional norm
is an if-then sentence (an implication) where the antecedent is descriptive and
the consequent is purely normative. Hence, a conditional norm has the form

\[ p \rightarrow H(q) \]

where p and q are descriptive sentences and H is a norm-creating operator, for
example Shall or May. As suggested above, it is possible to extend ordinary
propositional logic with propositional operators as Shall and May, etc. The
branch of logic derived in this way is called deontic logic. ‘Deontic’ comes from
the Greek word ‘deont’, which means “that which is binding”. Expressed in a
very general way, deontic logic is the logical study of obligation and permission.
The modern study of this kind of logic is often said to have commenced with
the article “Deontic Logic” by the Finnish philosopher Georg Henrik von Wright
published in *Mind* in 1951. This theory was anticipated by Ernst Mally in the 1920s and, much earlier, by Gottfried Wilhelm Leibniz (1646-1716) and Jeremy Bentham (1748-1832). The core of standard deontic logic is the formal study of the deontic operators ’it is permissible that’ (May) and ’it is obligatory that’ (Shall) and we can extend predicate logic as well as propositional logic with these operators.

1.3.2 Norms in predicate logic and as ordered pairs

A conditional norm is (usually) expressed as a universal sentence. For example:

(n1) For any \(x, y\) and \(z\) : if \(x\) has promised to pay \(\$y\) to \(z\), then \(x\) has an\nobligation to pay \(\$y\) to \(z\).

Within predicate logic, we can formalize (n1) as follows:

\((n_2) \forall x, y, z : \text{PromisedPay}(x, y, z) \rightarrow \text{Obligation_to_Pay}(x, y, z)\)

Thus, a typical conditional norm is a universal implication. Syntactically it consists of three parts: the sequence of universal quantifiers, the antecedent formula and the consequent formula. Note that the norm \((n_2)\) correlates open sentences: \(\text{PromisedPay}(x, y, z)\) is correlated to \(\text{Obligation_to_Pay}(x, y, z)\). A norm like \((n_2)\) can therefore be represented as a relational statement correlating a ground, \(\text{PromisedPay}\), to a consequence, \(\text{Obligation_to_Pay}\):

\[\text{PromisedPay} \in \mathcal{R} \text{ Obligation_to_Pay}.\]

Generally, \(p\mathcal{R}q\) represents the norm

\((n_3) \forall x_1, \ldots , x_n : p(x_1, \ldots , x_n) \rightarrow q(x_1, \ldots , x_n)\)

given that \(p\) and \(q\) are \(\nu\)-ary predicates. It is important here that the free variables in \(p(x_1, \ldots , x_n)\) are the same and in the same order as the free variables in \(q(x_1, \ldots , x_n)\). \(\mathcal{R}\) is a binary relation, and \(p\mathcal{R}q\) is a relational statement equivalent to \((p, q) \in \mathcal{R}\). Thus, a norm can be represented as \(p\mathcal{R}q\) or \((p, q) \in \mathcal{R}\). If, in the actual context, it can be tacitly understood and therefore omitted, it is only a small step to the representation of \((n_3)\) as the ordered pair \((p, q)\).

Note that \(p\mathcal{R}q\) as a representation of \((n_3)\) does not generally presuppose that \(q\) is a normative (or deontic) predicate, so \(p\mathcal{R}q\) can be used as a representation of any sentence which has the same form as \((n_3)\). Therefore, in many contexts of application the implicational relation \(\mathcal{R}\) can be such that only some of the sentences \(p\mathcal{R}q\) are norms. For reasons that will be explained when the formal framework is discussed, \(p\mathcal{R}q\) will be abbreviated as the ordered pair \((p, q)\) only when \(p\) and \(q\) are conditions of different sorts.

In the above discussion of the representation of norms, \(p\) and \(q\), as well as \(\text{PromisedPay}\) and \(\text{Obligation_to_Pay}\), appear as predicates. But the term predicate is often used for syntactical entities, and, therefore, interpreting \(p\mathcal{R}q\),

\(^2\)The development of deontic logic is closely related to another, better known part of logic, namely modal logic. The core of modal logic is the formal study of the operators ‘it is possible that’ and ‘it is necessary that’ (the so-called alethic modalities) and modal propositional logic is propositional logic extended with the possibility- and necessity-operator.
1.3. NORMATIVE SYSTEMS

![Diagram of potential consequences, joinings, and potential grounds]

Figure 1.1: A simple normative system.

$p$ and $q$ will here instead be conceived of as conditions. If $p$ is a $\nu$-ary condition and $i_1, \ldots, i_\nu$ are individuals, then $p(i_1, \ldots, i_\nu)$ is a statement. Antecedents and consequences of norms are represented as conditions and are called grounds and consequences respectively. A norm is represented as a statement relating (or correlating) a ground to a consequence, or represented as an ordered pair consisting of a ground and a consequence. In the preliminary analysis put forward in this section, grounds are descriptive and consequences are normative conditions.\(^3\)

Note that *Obligation to Pay* is a normative condition but that the sentence *Obligation to Pay*($x, y, z$) can be analysed as

OBLIGATORY Pay($x, y, z$).

where OBLIGATORY is a deontic operator resulting in a new predicate when it is applied to a given predicate. Pay is a descriptive condition and by applying the deontic operator OBLIGATORY we can in a sense construct a normative condition OBLIGATORY Pay out of the descriptive condition Pay. It is presupposed here that ‘OBLIGATORY Pay’ is equivalent to *Obligation to Pay* and I will return to this way of constructing normative conditions out of descriptive conditions using deontic operators.

Within the framework of the above preliminary analysis of norms, we can view a normative system $\mathcal{N}$ as consisting of a system $\mathcal{B}_1$ of potential grounds (descriptive conditions) and a system $\mathcal{B}_2$ of potential consequences (normative conditions). The set of norms in $\mathcal{N}$ are the set $J$ of links or joinings from $\mathcal{B}_1$ to $\mathcal{B}_2$. The Figure 1.1 is an attempt to illustrate the situation, where a norm is represented by an arrow from the system of grounds to the system of consequences.

A norm in a normative system $\mathcal{N}$, the norm here represented as an ordered pair $(p, q)$, can be regarded as a mechanism of inference. We can distinguish two

\(^3\)Cf. Odelstad & Lindahl (2002), pp. 32 ff and paper (1), section 3.2.
cases. Suppose that \( p \) and \( s \) are descriptive conditions and \( q \) and \( t \) normative. Then the following "derivation schemata" are valid given \( \mathcal{N} \).

1. 
\[
p(i_1, \ldots, i_{\nu}) \\
\langle p, q \rangle \\
q(i_1, \ldots, i_{\nu})
\]

2. 
\[
s \mathcal{R} p \\
\langle p, q \rangle \\
q \mathcal{R} t \\
\langle s, t \rangle \text{ } ^4
\]

In (1), \( \langle p, q \rangle \) functions as a deductive mechanism correlating sentences by means of instantiation, while in (2), \( \langle p, q \rangle \) plays an important role in correlating one condition, \( s \), to another condition, \( t \). \text{ } ^5

A condition, as the term is used here, is very similar to a relation; in a sense a condition is used for "expressing" a relation. \text{ } ^6 Relations, and therefore also conditions, are a specific kind of concepts. A normative system is thus a system consisting of an implicative relation between concepts. Note that the kind of normative systems we have encountered so far consists of two sorts of concepts, descriptive and normative.

Easily observable, conjunctions, disjunctions and negations of conditions can be formed by the operations \( \land, \lor, \lnot \), namely in the following way (where \( x_1, \ldots, x_{\nu} \) are place-holders, not individual constants).

\[
(p \land q)(x_1, \ldots, x_{\nu}) \text{ if and only if } p(x_1, \ldots, x_{\nu}) \text{ and } q(x_1, \ldots, x_{\nu}).
\]

\[
(p \lor q)(x_1, \ldots, x_{\nu}) \text{ if and only if } p(x_1, \ldots, x_{\nu}) \text{ or } q(x_1, \ldots, x_{\nu}).
\]

\[
(q')(x_1, \ldots, x_{\nu}) \text{ if and only if not } p(x_1, \ldots, x_{\nu}).
\]

\( \perp \) (Falsum) is the empty condition, not fulfilled by any \( \nu \)-tuple, and \( \top \) (Verum) is the universal condition, fulfilled by all \( \nu \)-tuples.

As is well-known, the truth-functional connectives can be used as operations in Boolean algebras. It is therefore possible to construct Boolean algebras of conditions. The role of the set of norms is to join two Boolean algebras:

- a Boolean algebra of grounds,

---

\text{ } ^4\text{Note that } s \mathcal{R} p \text{ relates conditions of the same sort and the same holds for } q \mathcal{R} t; s \text{ and } p \text{ are descriptive but } q \text{ and } t \text{ are normative. A norm consists of conditions of different sorts. As stated earlier, only implicative sentences that relate conditions of different sorts will be represented as ordered pairs.}

\text{ } ^5\text{See paper (1) subsection 3.2 and paper (2) subsection 2.2. Cf. Alchourrón & Bulygin (1971) p. 28. Schema 1 corresponds to what Alchourrón and Bulygin call the correlation of individual cases to individual solutions, and schema 2 corresponds to what they call the correlation of generic cases to generic solutions.}

\text{ } ^6\text{Properties are here regarded as unary relations and can be "expressed" by conditions.}
• a Boolean algebra of consequences.

The norms are links or joinings between the algebra of grounds and the algebra of consequences.

The outline of the algebraic approach to normative systems just presented is substantially simplified. The approach will be developed extensively below.

1.4 Conceptual systems

In the previous subsection, a simple normative system has been characterized as a two-sorted implicative conceptual system, where the concepts are, from a logical point of view, relations (expressed as conditions) and the two sorts of concepts involved are descriptive and normative conditions. However, relations (and therefore also conditions) are only one specific kind of concepts, where ‘kind’ is something else than ‘sort’. Other kinds of concepts are, inter alia, aspects (in philosophy of science often called attributes) and measures (often termed scales). Examples of aspects are length, weight, temperature, intelligence, utility and probability. Examples of measures are meter, kilogram, degrees centigrade and the probability measure. Different kinds of concepts have different logical form (for example relations, structures and functions) while different sorts of concepts differ in their cognitive status (for example descriptive and normative respectively). From a logical point of view, aspects are structures and measures are functions.

When studying implicative conceptual systems where the concepts are conditions, the implicative relation is implication in a straightforward sense. However, when the concepts are aspects or scales, we are dealing with implicative relations that are implications only in a rather generalized sense. Implicative statements, i.e. statements expressing that an implicative relation holds, can in such cases, for example, be interpreted as determination or relevance. However, we will even in the generalized contexts talk about the antecedent and consequent of an implicative statement, and even of grounds and consequences.

As pointed out above, for concepts which are conditions we can in an obvious way define the operations conjunction, disjunction and negation and thereby arrive at a Boolean algebra. The situation is different for concepts that are aspects, since taking the negation of an aspect is not certainly a meaningful operation. However, aspects can form a lattice. We shall discuss this further in chapter 2.

‘Concept’ is a complicated notion and is of great importance in many areas. It is tightly connected to the notion of ‘meaning’, and ‘the meaning of concepts’ is a philosophical minefield. But in this context, it is impossible to avoid the term ‘concept’. The following short passage from the entry Concept in The Encyclopedia of Philosophy describes its usefulness:

Concept is one of the oldest terms in the philosophical vocabulary, and one of the most equivocal. Though a frequent source of confusion and controversy, it remains useful, precisely because of its ambiguity,
as a sort of passkey through the labyrinths represented by the theory
of meaning, the theory of thinking, and the theory of being. (Heath,
1967.)

In this thesis, the use of the notions ‘concept’ and ‘meaning’ is instrumental, and
these notions function as passkeys to the main objectives of the work presented
here. As the word ‘concept’ is used, complex combinations of concepts are still
regarded as concepts. A concept can be defined in terms of other concepts in a
more or less complicated way.

A notion connected to ‘concept’ that will play a role in the thesis is ‘cognitive
status’. The idea is that the different sorts of concepts constituting an msic-
system are often different with respect to their cognitive status. As a source
of inspiration for using the notion ‘cognitive status’ in the theory of msic-systems
one can take Ernest Nagel’s discussion of the cognitive status of scientific theo-
ries in his book *The Structure of Science*. But here the term ‘cognitive status’
is applied to concepts. Examples of different cognitive status include: logical,
empirical, observational, operational, theoretical, physical, mental, descrip-
tive, prescriptive, normative, evaluative, and—as we will see below—intermediate.
(Note that several of the different sorts of cognitive status exemplified above can
be applied to the same concepts.)

1.5 Intermediate concepts—form and function

1.5.1 Intermediaries

In the simplified presentation in section 1.3, a normative system is represented
as a two-sorted implicational conceptual system, consisting of a set of descriptive
grounds and a set of normative consequences. However, many concepts for
example in law are neither purely descriptive nor purely normative. Like Janus,
the Roman god of beginnings and endings, they have two faces, one turned
towards facts and description, the other towards legal consequences. These
concepts are said to be intermediate between facts and legal consequences and
will often be called intermediaries. Figure 1.2 will give a first illustration of this
idea.

As an example, consider what it means to be a citizen according to the
system of the U.S. Constitution. Article XIV, section 1 reads as follows:

 All persons born or naturalized in the United States, and subject
to the jurisdiction thereof, are citizens of the United States and of
the State wherein they reside. No State shall make or enforce any
law which shall abridge the privileges or immunities of citizens of the
United States; nor shall any State deprive any person of life, liberty,
or property, without the due process of law; nor deny to any person
within its jurisdiction the equal protection of the laws.

Two key concepts in the article are *citizen* and *person*. The article specifies
the ground for the condition being a citizen in the United States:
1.5. *INTERMEDIATE CONCEPTS—FORM AND FUNCTION*

![Figure 1.2: A normative system with intermediaries.](image)

persons born or naturalized in the United States, and subject to the jurisdiction thereof

and specifies a number of regal consequences of this condition expressed in terms of ‘shall’:

no State shall make or enforce any law which shall abridge the privileges or immunities of citizens of the United States.

The article does not state any ground for the condition to be a person but specifies a number of legal consequences connected to this condition:

nor shall any State deprive any person of life, liberty, or property, without due process of law; nor deny to any person within its jurisdiction the equal protection of the laws.

Within the constitutional system of United States, this article is supplemented with rules laid down by the Constitution and through court decisions. These rules determine together, by specifying grounds and consequences, the role the concept ‘citizen’ and ‘person’ have within the legal system.

Let us construct a simplified “condition-implicative” representation of the legal rules described above. According to the rules, the disjunction of the two conditions

\[ b: \text{to be a person born in the U.S.} \]
\[ n: \text{to be a person naturalized in the U.S.} \]

in conjunction with the condition

---

7 The concept citizen regarded as an intermediary is discussed in Odestad & Lindahl (1998), Odestad & Lindahl (2000) and Lindahl & Odestad (2000). In paper (3), citizenship is treated from the point of view of organic wholes.
s: to be a person subject to the jurisdiction of the U.S.
implies the condition
c: to be a citizen of the U.S.

That this implicative relationship holds according to the system is represented in the form \((b \lor n) \land s) \rightarrow Rc\). Since it is a settled matter that citizens who are minors do not have the right to vote in general elections, c does not imply the condition
e: to be entitled to vote in general elections.

Therefore: not \([cRc]\), and hence not \(((b \lor n) \land s) \rightarrow Rc\).

Let
a: to be adult.

Simplifying matters, suppose that,

\[ (c \land a) \rightarrow Rc. \]

It is easy to see that this is equivalent to

\[ cR(a' \lor e). \]

Going from (1) to (2) can be called exportation, and going from (2) to (1) importation.

We thus have within the system the following rules: \((b \lor n) \land s) \rightarrow Rc\) and \(cR(a' \lor e)\), stating that the condition \((b \lor n) \land s\) is a ground for c and \((a' \lor e)\) is a consequence of c. These two rules determine partly the role of c (citizenship) in the constitutional system under study. But there can also be other grounds \(g_1, g_2, \ldots\) of c and consequences \(h_1, h_2, \ldots\) of c within the constitutional system. Suppose that \(g_1, g_2, \ldots\) are the grounds of c and \(h_1, h_2, \ldots\) the consequences of c. Hence, the role of c in the system is characterized by

\[ g_1Rc, g_2Rc, \ldots, cRh_1, cRh_2, \ldots \]

The concept c thus couples a set of legal consequences to a set of legal grounds and c is situated “intermediate” between the set of grounds and the set of consequences. Concepts of this kind are called intermediate concepts or intermediaries. Over the past sixty years, there has been an on-going discussion in Scandinavia as regards the idea of intermediate concepts in the law. The debate was started in 1944-1945 by Anders Wedberg and Per-Olof Ekelöf, and in 1951 Alf Ross published his well-known essay on “Tú-Tú”. In this debate, an often used example is the concept of ownership. Ross represents a set of legal rules concerning ownership (denoted O) in essentially the following way, where \(F_i\) expresses a possible legal ground and \(C_j\) a legal consequence of O.

\[ \text{For a more detailed analysis of the early Scandinavian debate see Lindahl & Odelstad (1998a) section 1.2.} \]
Ross himself comments on this scheme in the following way:

“O” (ownership) merely stands for the systematic connection that \( F_1 \), as well as \( F_2, F_3, \ldots, F_p \) entail the totality of legal consequences \( C_1, C_2, C_3, \ldots, C_n \). As a technique of presentation this is expressed then by stating in one series of rules the facts that “create ownership” and in another series the legal consequences that “ownership” entails. (Ross 1956-57, p. 820.)

Note that the rules that “create ownership” can be expressed by one rule:

\[
F_1 \lor \ldots \lor F_p \rightarrow O. \quad (9)
\]

And the rules describing what ‘ownership’ entail can also be condensed to one rule:

\[
O \rightarrow C_1 \land \ldots \land C_n.
\]

So an equivalent way of representing the legal rules concerning ownership according to Ross is the following scheme:

\[
F_1 \lor \ldots \lor F_p \rightarrow O \rightarrow C_1 \land \ldots \land C_n
\]

Whereas \( F_1, \ldots, F_p \) can be called grounds and \( C_1, \ldots, C_n \) consequences of \( O \), \( F_1 \lor \ldots \lor F_p \) is the ground of \( O \) and \( C_1 \land \ldots \land C_n \) the consequence of \( O \).

Note that the rule \( F_i \rightarrow O \) is a way of introducing \( O \) into the discourse, and appropriately we can call such a rule an introduction rule of \( O \). In harmony with this, the rule \( O \rightarrow C_j \) can be called an elimination rule of \( O \), since in a sense such a rule can eliminate \( O \) from the discourse. Analogous to the use of the phrases ‘the ground’ and ‘the consequence’ we can say that

\[
F_1 \lor \ldots \lor F_p \rightarrow O
\]

is the introduction rule of \( O \) and

\[
O \rightarrow C_1 \land \ldots \land C_n
\]

is the elimination rule of \( O \). Introduction and elimination rules are discussed in paper (4) with reference to Gentzen.

In Wedberg (1951), three different methods for treating the concept of ‘ownership’ are discussed. The first and second of these methods aim at a definition of ownership in terms of grounds and consequences respectively. Wedberg’s third method treats ownership as a ‘vehicle of inference’. According to Wedberg this means that ownership is a tool for inferring statements of legal consequences

\( ^9 \) is a consequence relation.
from statements of legal facts, and, therefore, ownership is undefined. Obviously, Wedberg’s third method for treating ownership is close to Ross’s view.

We will return to the question of defining intermediate concepts in relation to regarding them as vehicles of inferences in the next chapter. As a point of departure for further discussions and refinements, we regard intermediate concepts as characterized by their grounds and consequences. The characterization of the concept citizenship, c, thus has the following form:

\[ g_1 Rc, g_2 Rc, ..., cRh_1, cRh_2, ... \]

For the view of intermediate concepts developed in this thesis, the discussion in legal philosophy has been an important source of inspiration. But there are of course also other theories that have influenced this research. The following quotation from Lindahl & Odelstad (1999a) emphasizes this, where “the ideas mentioned above” are the ideas of Wedberg and Ross.

In the theory of language of Michael Dummett, there are features with some resemblance to the ideas mentioned above. According to Dummett, the meaning of an expression is determined, on one hand by the condition for correctly uttering it, and on the other hand by what the uttering of the expression commits the speaker to. Therefore, the meaning of a statement is identified in part by the conditions from which it can be inferred and in part by what can be inferred from the statement. In the case of utterances of sentences composed by the connectives “and”, “or” etc., this is given by what are called introduction and elimination rules in Gentzen’s system of natural deduction. (Lindahl & Odelstad, 1999a, p. 165.)

Introduction and elimination rules are discussed in paper (4).

The analysis of the concept of ‘intermediary’ involves complicated questions of meaning and is therefore a philosophically loaded topic. In the papers the problems of meaning are discussed only briefly. The formal theory of intervenients which is presented in papers (4) and (5) is intended as a means for a thorough analysis of the concept of an intermediary.

An interesting issue in the discussion of intermediaries is the negation of an intermediate concept. Suppose that \( a_1 \) is the ground of the intermediary \( m \) and that \( a_2 \) is the consequence of \( m \). Let \( m' \) be the negation of \( m \), i.e. \( \neg m \). Is \( m' \) an intermediate concept? If the answer is yes, what can be said about its grounds and consequences? This question, which is discussed in paper (4), is complicated, especially if we turn to open intermediaries.

1.5.2 Open intermediaries

The concept ‘work of equal value’ is an essential concept in the Swedish Equal Opportunities Act. The following quotation demonstrates this (emphasis added here):
Employers and employees shall cooperate in pursuing active efforts to promote equality in working life. They shall strive in particular to prevent and eliminate differences in pay and in other conditions of employment between women and men performing work that may be considered equal or of equal value. They shall also promote equal opportunities for wage growth for women and men.

Work is to be considered equal in value to other work if, based on an overall assessment of the nature of the work and the requirements imposed on the worker, it may be deemed to be of similar value. Assessments of work requirements shall take into account criteria such as knowledge and skills, responsibility and effort. When the nature of the work is assessed, particular regard shall be taken of the working conditions.

The concept ‘work of equal value’ is an intermediary with—using the Janus-metaphor—one face looking at the nature of and requirements for the work and the other face looking at efforts to promote equality in working life, especially equal pay for equal work. The law does not supply us with a complete set of introduction rules for the concept. Instead it mentions some criteria that equality of value depends on, viz. knowledge and skills, responsibility and effort. However, one can extract the following uncontroversial introduction rule: if \( x \) and \( y \) are work that requires the same degree of knowledge, skills, responsibility and effort, then \( x \) and \( y \) are work of equal value. We can express this in a formalised style as follows:

\[
x \sim_1 y \land x \sim_2 y \land x \sim_3 y \land x \sim_4 y \land x \sim_5 y \quad \longrightarrow \quad x \sim_v y
\]

where

\( \sim_1 \) is the relation ‘equal knowledge’
\( \sim_2 \) is the relation ‘equal skills’
\( \sim_3 \) is the relation ‘equal responsibility’
\( \sim_4 \) is the relation ‘equal effort’
\( \sim_5 \) is the relation ‘equal working conditions’
\( \sim_v \) is the relation ‘equal value’

Note that the equality relations \( \sim_1, \sim_2, \sim_3, \sim_4 \) and \( \sim_5 \) are here regarded as conditions and we can therefore apply Boolean operations on the equality relations, for example construct conjunctions of them. One of the grounds of \( \sim_v \) is thus the condition

\[
\sim_1 \land \sim_2 \land \sim_3 \land \sim_4 \land \sim_5 .
\]

But it is also possible that work \( x \) and \( y \) are of equal value even if they are not equal with respect to the requirements knowledge, skills, responsibility, effort and working condition. We can imagine a situation such that \( x \) requires more knowledge than \( y \), and \( y \) more responsibility than \( x \) but that these two differences balance out. But to turn this observation into an introduction rule is often not possible. The applicability of the concept work of equal value in a
certain case must therefore be based on judgments of what holds in the actual case. And even if the law does not state detailed rules for these judgments it gives guidelines, for example in terms of what are possible inputs in such judgments or what factors or circumstances must be taken into account.

The grounds of the concept ‘work of equal’ value is thus only partially determined by the law in the form of introduction rules. The application of the concept in special cases deserves interpretative decisions based on the role and function of the concept in the law. We call such intermediaries ground-open. Concepts such that the consequences are only partially determined by elimination rules are called consequence-open.

Open intermediaries are further discussed in paper (4). For a detailed discussion of the concept work of equal value, see Odelstad (2008).

1.5.3 Intermediaries in normative systems

A normative system is only in rather special cases a two-sorted implicative conceptual system, i.e. a system of grounds and a system of consequences. Instead normative systems often contain also many intermediate concepts. In more complex normative systems, for example legal systems, there are usually more than one system of intermediaries, and these systems often form a kind of network, where between intermediaries of two different sorts there are intermediaries of a third sort. In paper (5), this is illustrated as Figure 1. There the lines between different nodes represent sets of introduction or elimination rules. Note that a rule can simultaneously be an introduction rule for one concept and an elimination rule for another.

Intermediaries do not only exist in normative systems but in many other msic-systems. This is discussed in Lindahl & Odelstad (1999a) p. 178.

1.5.4 Implicative closeness between strata

One important problem area in the study of msic-systems is the “closeness” between different strata. Some of the ideas regarding this topic will be informally described here.

Consider the norms (links) from the system $B_1$ of grounds to the system $B_2$ of consequences. One norm can be “narrower” than another, which is illustrated in Figure 1.3. Suppose that $(a_1, a_2)$ and $(b_1, b_2)$ are norms from the system of grounds $B_1$ to the system of consequences $B_2$.

Figure 1.3 illustrates that $(a_1, a_2)$ is narrower than $(b_1, b_2)$. We can say alternatively that $(a_1, a_2)$ “lies between” $b_1$ and $b_2$. We define the relation ‘at least as narrow as’, expressed by $\leq$, in the following way:

$$\langle a_1, a_2 \rangle \leq \langle b_1, b_2 \rangle \text{ if and only if } b_1 R a_1 \text{ and } a_2 R b_2.$$  

It is easy to see that if $R$ is a quasi-ordering, i.e. transitive and reflexive, then $\leq$ is also a quasi-ordering.

\cite{98See paper (3) p. 84.
A norm that is maximally narrow is minimal with respect to the relation ‘at least as narrow as’. Hence, a norm \( \langle a_1, a_2 \rangle \) is maximally narrow if there is no norm in the system that is strictly narrower than \( \langle a_1, a_2 \rangle \), i.e. if \( \langle a_1, a_2 \rangle \) is a minimal element with respect to ‘at least as narrow as’. In a normative system, the set of norms that are maximally narrow play a crucial role. Given certain requirements of a well-formed normative system, all the other norms of the system are determined by its maximally narrow norms and, therefore, any change of such a system implies a change of at least one maximally narrow norm.\(^{11}\) This is discussed in papers (3) and (4).

The idea behind intermediaries is that they are intermediate between different strata of concepts and offer narrow links between the strata. It is important to notice that the intermediaries between two strata constitute a strata itself. The introduction rules of the intermediaries are links from the “bottom strata” to the “intermediate strata” and the elimination rules of the intermediaries are links from the “intermediate strata” to the “top strata”. The introduction rule and the elimination rule of an intermediary constitute narrow links, since the introduction rule determines the weakest ground of the intermediary and the elimination rule the strongest consequence. Intermediate concepts are thus studied in terms of how narrow they are the structure of grounds and the structure of consequences. Generally, the “implicative closeness”\(^{\dagger}\) between strata is analysed using concepts as minimal joining, weakest ground and strongest consequence. Figure 1.4 illustrates the two last mentioned notions: \( a_1 \) is a weakest ground of \( m \) if \( b_1 \triangleright m \) implies \( a_1 \triangleright b_1 \). And \( a_2 \) is a strongest consequence of \( m \) if \( m \triangleright b_2 \) implies \( a_2 \triangleright b_2 \). As a preliminary approximation we can say that the introduction rule of an intermediary states its weakest ground and the elimination rule states its strongest consequence. In paper (5) this is discussed in more detail and a rudimentary typology of intermediate concepts is established.

\(^{11}\)See Odelstad & Lindahl (2002), p. 36.
1.6 Deontic consequences

Let us for a moment return to the simple picture of a normative system consisting of a system of grounds and a system of consequences. The consequences are normative conditions. So far, what we have said about normative conditions is just that they can be constructed by applying a deontic operation to descriptive conditions. There is an extensive literature on deontic operations and it is not intended to enter this discussion here but for the actual applications a well-established theory will be used. In this thesis, the combination of deontic and action logic developed by Stig Kanger will be used, especially the theory of normative positions created by Kanger and Lindahl. This theory is very suitable for the research described here.

1.6.1 Deontic logic with the action operator Do

Kanger exploited the possibilities of combining the deontic operator Shall with the binary action operator Do. The operation Do means that one sees to it that something is the case (see Kanger, 1957). To be more exact, Shall Do(x, q) means that it shall be that x sees to it that q, while for example ¬Shall Do(y, ¬q) means that it is not the case that it shall be that y sees to it that not q. The combination of the deontic operator Shall with the action operator Do and the negation operation ¬ gives us a powerful language for expressing purely normative sentences. Kanger emphasized the possibilities of external and internal negation of sentences where these operators are combined. Using combinations of deontic and action operators, we can formulate norms in a more effective way. A conditional norm may for example have the following form: ‘If p then it shall be the case that x sees to it that q’, which thus can be written as
1.6. DEONTIC CONSEQUENCES

\[ p \rightarrow \text{Shall Do}(x, q). \]

In such norms, \( p \) is often a state of affairs which is about \( x \) and \( y \), while \( q \) is a state of affair which deals with \( y \), i.e. \( p \) can be seen as predicate with \( x \) and \( y \) as variables while \( q \) is a predicate with \( y \) as the only variable. Hence, a conditional norm can have the following form:

\[ p(x, y) \rightarrow \text{Shall Do}(x, \neg q(y)). \]

A concrete example of a norm which has this form is as follows. Suppose that \( p(x, y) \) means that \( x \) owns \( y \) and \( y \) is a dog while \( q(y) \) means that \( y \) fouls public places. The norm above then says that the owner of a dog shall see to it that the dog does not foul in public places.

Note that the sentence May Do\((x, q)\) can be defined in terms of the operators Shall and Do in the following way:

\( \text{May Do}(x, q) \) if and only if \( \neg \text{Shall} \neg \text{Do}(x, q) \).

It is worth noting that conditional norms have some similarities with production rules. According to Luger (2002) p. 171, a production rule is

\[ \text{a condition-action pair} \quad \text{defines a single chunk of problem-solving knowledge. The condition part} \quad \text{of the rule is a pattern that determines when that rule may be applied to a problem instance. The action part} \quad \text{defines the associated problem-solving step.} \]

The antecedent (or ground) in a norm corresponds to the condition part in a production rule, and the consequent (or consequence) in a norm corresponds to the action part. A production rule thus has the logical form

\[ p \rightarrow \text{Do } q \]

or perhaps better

\[ p \rightarrow \text{Shall Do } q. \]

1.6.2 Normative positions

In 1913, the American jurist Wesley Newcomb Hohfeld published a work in philosophy of law which has been very influential. It carries the title *Fundamental Legal Conceptions as Applied in Judicial Reasoning* and contains a characterization of eight fundamental legal notions, which were meant to serve as fundamental elements in the analysis of more complex legal relations. Inspired by Hohfeld’s work, Kanger developed a theory of normative positions using the deontic-action-language. Kanger’s theory of normative positions was originally expressed as a theory of types of rights. He emphasized that the term ‘right’ has various meanings. For example, if Mrs. \( x \) has lent 100 dollars to Mr. \( y \), then \( x \) has a right of the simple type Claim against \( y \) that she gets back the money she has lent to \( y \). Let

\[ q_1(x, y) : \text{\( x \) gets back the money \( x \) has lent to \( y \).} \]

The type of right Claim with regard to \( q_1(x, y) \) is defined in the following way:
Claim\((x, y, q_1(x, y))\) if and only if \(\text{Shall } Do(y, q_1(x, y))\).

This means that \(y\) shall see to it that \(x\) gets back the money she lent \(y\). Further, Mrs. \(x\) has probably a right of type Immunity to walk outside Mr. \(y\)'s shop. Let

\[ q_2(x, y) : x \text{ walks outside } y \text{'s shop} \]

Immunity with regard to \(q_2\) is defined as follows:

\[ \text{Immunity}(x, y, q_2(x, y)) \text{ if and only if } \text{Shall } \neg \text{Do}(y, \neg q_2(x, y)). \]

Hence, it shall be the case that \(y\) does not see to it that \(x\) does not walk outside \(y\)'s shop. (These examples are taken from Lindahl, 1994, p. 891-892.)

Kanger’s work was considerably improved and extended into a formal theory of normative positions in Lindahl (1977). Lindahl developed three systems of types of normative positions. The simplest one is the system of one-agent types of normative position, and only this system is used in this thesis.\(^\text{12}\) The one-agent types are constructed in the following way. Let \(+\alpha\) stand for either of \(\alpha\) or \(-\alpha\). Starting from the scheme \(\pm \text{May} \pm \text{Do}(x, \pm q)\), where \(\pm\) stands for the two alternatives of affirmation or negation, a list is made of all maximal and consistent conjunctions, ‘maxiconjunctions’, such that each conjunct satisfies the scheme.\(^\text{13}\) Maximality means that if we add any further conjunct, satisfying the scheme, then this new conjunct either is inconsistent with the original conjunction or redundant. Note that the expression \(\neg \text{Do}(x, q) \& \neg \text{Do}(x, \neg q)\) expresses \(x\)’s passivity with regard to \(q\). Here this expression is abbreviated as Pass\((x, q)\). By this procedure, the following list of seven maxiconjunctions is obtained, which are denoted \(T_1(x, q), \ldots, T_7(x, q)\), see Lindahl (1977), p. 92.

\[ T_1(x, q) : \text{MayDo}(x, q) \& \text{MayPass}(x, q) \& \text{MayDo}(x, \neg q). \]

\[ T_2(x, q) : \text{MayDo}(x, q) \& \text{MayPass}(x, q) \& \neg \text{MayDo}(x, \neg q). \]

\[ T_3(x, q) : \text{MayDo}(x, q) \& \neg \text{MayPass}(x, q) \& \text{MayDo}(x, \neg q). \]

\[ T_4(x, q) : \neg \text{MayDo}(x, q) \& \text{MayPass}(x, q) \& \text{MayDo}(x, \neg q). \]

\[ T_5(x, q) : \text{MayDo}(x, q) \& \neg \text{MayPass}(x, q) \& \neg \text{MayDo}(x, \neg q). \]

\[ T_6(x, q) : \neg \text{MayDo}(x, q) \& \text{MayPass}(x, q) \& \neg \text{MayDo}(x, \neg q). \]

\[ T_7(x, q) : \neg \text{MayDo}(x, q) \& \neg \text{MayPass}(x, q) \& \text{MayDo}(x, \neg q). \]

\(T_1, \ldots, T_7\) are called the types of one-agent positions.\(^\text{14}\) Given the underlying logic, the one-agent types are mutually disjoint and their union is exhaustive, i.e. constitute a partition. Note that \(\neg \text{MayDo}(x, q) \& \neg \text{MayPass}(x, q) \& \neg \text{MayDo}(x, \neg q)\) is logically false, according to the logic of Shall and May.

\(^{\text{12}}\)This system is presented in paper (1) and (2), but it will also be presented here since it is presupposed in the section about applications of msic-systems.

\(^{\text{13}}\)The notion of ‘maxiconjunction’ was introduced in Makinson (1986), p. 405f.

\(^{\text{14}}\)Formally, a “type” \(T_i\) (1 \(\leq\) \(i\) \(\leq\) 7) of one-agent positions refers to the set of all ordered pairs \((x, q)\) such that \(T_i(x, q)\).
1.6. **Deontic Consequences**

It is easy to see that the last three types can more concisely be described as follows:

\[ T_5(x, q) : \text{Shall Do}(x, q). \]

\[ T_6(x, q) : \text{Shall Pass}(x, q). \]

\[ T_7(x, q) : \text{Shall Do}(x, \neg q). \]

Note that the following “symmetry principles” hold (Lindahl, 1977, p. 92):

\[ T_1(x, q) \text{ if and only if } T_1(x, \neg q) \]

\[ T_2(x, q) \text{ if and only if } T_4(x, \neg q) \]

\[ T_3(x, q) \text{ if and only if } T_3(x, \neg q) \]

\[ T_5(x, q) \text{ if and only if } T_7(x, \neg q) \]

\[ T_6(x, q) \text{ if and only if } T_6(x, \neg q) \]

In papers (1) and (2) the one-agent-types in the Kanger-Lindahl theory of normative positions are used as operators on descriptive conditions to get deontic conditions. As a simple example, suppose that \( r \) is a unary condition. Then \( T_i r \) (with \( 1 \leq i \leq 7 \)) is the binary condition such that

\[ T_i r(y, x) \text{ iff } T_i(x, r(y)), \]

where \( T_i(x, r(y)) \) is the \( i \)th formula of one-agent normative positions. Note that for example \( T_3(x, r(y)) \) means

\[ \text{MayDo}(x, r(y)) \land \neg \text{MayPass}(x, r(y)) \land \text{MayDo}(x, \neg r(y)). \]

\( T_i \) is called a one-agent position-operator. If \( \langle p, T_i r \rangle \) is a norm, then from \( p(x_1, x_2) \) we can, by using the norm, infer \( T_i r(x_1, x_2) \) and thus also \( T_i(x_2, r(x_1)) \), which means that, with regard to the state of affairs \( r(x_1) \), \( x_2 \) has a normative position of type \( T_i \).

The theory of normative positions was developed during the 60s and 70s, primarily as an analytical tool to be used in jurisprudence and political science. The Kanger-Lindahl theory of normative positions was applied to problems in computer science in the 90s, see Jones & Sergot (1993) and (1996), Sergot (1999), Krogh (1995) and Krogh & Herrestad (1999).

### 1.6.3 Normative systems as msic-systems

Conceiving of normative systems as msic-systems is a kind of representation of normative systems. What characterizes the subclass of normative systems among msic-systems in general are their cognitive features. A normative system consists of one stratum of descriptive grounds and another stratum of normative consequences and eventually one or more strata of intermediaries. Furthermore,
a normative system contains links or joinings between the strata. Note that the final consequences are expressed in terms of normative conditions, for example constructed by applying deontic operations to descriptive conditions. Thus, representing normative systems in this way puts the emphasis on concepts and not on propositions.

1.7 The algebraic approach to \textit{msic}-systems

The study of the structure of \textit{msic}-systems, especially the implicative closeness between different strata, is one of the main goals of this thesis. As tools for this endeavour, algebraic concepts and theories are used. In this section, two of the structures that play a crucial role in the thesis will be described briefly. But first a preliminary remark.

1.7.1 Set-theoretical predicates

A common way of characterizing formal theories in mathematics is described by Suppes as follows:

The kernel of the procedure for axiomatizing theories within set theory may be described very briefly: to axiomatize a theory is to define a predicate in terms of notions of set theory. A predicate so defined is called a \textit{set-theoretical} predicate. (Suppes, 1957, p. 249.)

A simple example of a set-theoretical predicate is ‘to be a quasi-ordering’:

**Definition 1** Let $A$ be a set and $R$ a binary relation on $A$. The relational structure $(A, R)$ is a quasi-ordering if for all $a, b, c$ in $A$, the following axioms are satisfied:

1. $aRa$ (reflexivity)
2. If $aRb$ and $bRc$, then $aRc$ (transitivity).

‘To be a quasi-ordering’ is a predicate, which is true or false of relational structures. This set-theoretical predicate characterizes an axiomatized theory, the theory of quasi-orderings, and a model of that theory is a structure satisfying the predicate ‘to be a quasi-ordering’.

Two set-theoretical predicates which play a crucial role in the thesis will now be presented.

1.7.2 Boolean quasi-orderings and joining systems

**Definition 2** The relational structure $(B, \land, \lor, R)$ is a Boolean quasi-ordering (Bqo) if $(B, \land, \lor)$ is a Boolean algebra, $R$ is a quasi-ordering, $\perp$ is the zero element, $\top$ is the unit element and $R$ satisfies the additional requirements:

1. $aRb$ and $aRc$ implies $aR(b \land c)$,
2. $aRb$ implies $b \lor Ra$,
1.7. THE ALGEBRAIC APPROACH TO MSIC-SYSTEMS

(3) \((a \land b)Ra\),
(4) \(\not\top R1\).

Boolean algebras are well-known structures with many applications. A Boolean quasi-ordering is a quasi-ordering defined on a Boolean algebra in such a way that it determines a new Boolean algebra related to the first one in a special way. This is explained in more detail in paper (1). The definition of a Boolean joining system, which follows below, presupposes the definition of a Boolean quasi-ordering. Many normative systems can be represented as Boolean joining systems or combinations of two or more such systems. First a reminder of a notion discussed earlier:

**Definition 3** The narrowness-relations determined by the quasi-orderings \(\langle B_1, R_1 \rangle\) and \(\langle B_2, R_2 \rangle\) is the binary relation \(\preceq\) on \(B_1 \times B_2\) such that \(\langle a_1, a_2 \rangle \preceq \langle b_1, b_2 \rangle\) if and only if \(b_1 R_1 a_1\) and \(a_2 R_2 b_2\).

Note that \(\preceq\) is a quasi-ordering on \(B_1 \times B_2\).

**Definition 4** A Boolean joining system \((Bjs)\) is an ordered triple \(\langle B_1, B_2, J \rangle\) such that \(B_1 = \langle B_1, \land', R_1 \rangle\) and \(B_2 = \langle B_2, \land', R_2 \rangle\) are Bqo’s and \(J \subseteq B_1 \times B_2\), and the following requirements are satisfied:
(1) for all \(b_1, c_1 \in B_1\) and \(b_2, c_2 \in B_2\), \(\langle b_1, b_2 \rangle \in J\) and \(\langle b_1, b_2 \rangle \preceq \langle c_1, c_2 \rangle\) implies \(\langle c_1, c_2 \rangle \in J\),
(2) for any \(C_1 \subseteq B_1\) and \(C_2 \subseteq B_2\), if \(\langle c_1, b_2 \rangle \in J\) for all \(c_1 \in C_1\), then \(\langle a_1, b_2 \rangle \in J\) for all \(a_1 \in \text{lub}_{B_1} C_1\),
(3) for any \(C_2 \subseteq B_2\) and \(b_1 \in B_1\), if \(\langle b_1, c_2 \rangle \in J\) for all \(c_2 \in C_2\), then \(\langle b_1, a_2 \rangle \in J\) for all \(a_2 \in \text{glb}_{B_2} C_2\).

A norm can, as has been pointed out above, in many contexts be regarded as consisting of two objects, a ground condition and a consequence condition standing in an implicative relation to each other. The ground belongs to one Boolean quasi-ordering and the consequence to another. Therefore, we can view a normative system as a set of joinings of a Boolean quasi-ordering of grounds to a Boolean quasi-ordering of consequences, where \(\land\) and \(\lor\) are Boolean operations on the conditions. A normative system \(N\) can therefore be represented as a Boolean joining system \(\langle B_1, B_2, J \rangle\) where \(B_1 = \langle B_1, \land', R_1 \rangle\) is a Boolean quasi-ordering of ground-conditions, \(B_2 = \langle B_2, \land', R_2 \rangle\) a Boolean quasi-ordering of consequence-conditions and the set \(J\), where \(J \subseteq B_1 \times B_2\), is the set of norms. Note that the implicational relation in the system \(N\) is represented in the different parts of the system by the relations \(R_1, R_2\) and \(J\) respectively.

It is worth noting there is a difference in notational conventions between the definition of a Bqo and the definition of a Bjs. In a Bqo, if the relation \(R\) holds between \(a\) and \(b\) this is written \(aR b\). If in a Bjs \(J\) holds between \(a_1\) and \(a_2\) this is written \(\langle a_1, a_2 \rangle \in J\). The reason is that in the intended models of Bjs’s, the elements in \(J\) are treated as objects in a way that does not hold for the elements in \(R\). In a representation of a Bjs as a normative system, \(\langle a_1, a_2 \rangle \in J\)
means that the norm \( \langle a_1, a_2 \rangle \) holds in the system, and the elements in \( J \) are subject to comparison with respect to, for example, narrowness.

Given the narrowness relation \( \preceq \) one can determine the set of minimal elements of \( J \), \( \text{min}\, J \), with respect to \( \preceq \). Under fairly general conditions (specified in the papers), the set \( \text{min}\, J \) characterizes \( J \) in the following way:

\[
\langle a_1, a_2 \rangle \in J \quad \text{iff} \quad \exists \langle b_1, b_2 \rangle \in \text{min}\, J : \langle b_1, b_2 \rangle \preceq \langle a_1, a_2 \rangle.
\]

Given certain general presuppositions, one can choose a subset \( C \) of \( \text{min}\, J \) from which \( \text{min}\, J \) can be inferred and which therefore also determines \( J \). We call such a set \( C \) a base of minimal elements of \( J \). In many contexts, the elements in \( C \) can be represented by intermediate concepts. An intermediary is determined by the condition that constitute its maximally narrow ground and the condition that constitutes its maximally narrow consequence. See papers (4) and (5) for further details.

### 1.7.3 Models and variations of the algebraic theories

As has been emphasized in earlier sections with normative systems as a key example, one approach to the representation of msic-systems is by regarding concepts as conditions subject to Boolean operations and with an implicative relation defined on these conditions. A \( Bqo \) or a \( Bjs \) with domains of conditions is called a condition implication structure, abbreviated \( cis \). A special kind of \( cis \)-representation of a normative system is the \( npcis \)-representation of normative conditions. In an \( npcis \), a normative condition is constructed by applying the one-agent position-operators to descriptive conditions. See paper (1).

There are some limitations of the \( cis \)-representation of \( msic \). One problem is the formation of conjunctions and disjunctions of conditions of different arity. How this can be handled is discussed in the next chapter, where the problem of the tacitly assumed order of the variables in the conditions also will be observed. Another weakness of the \( cis \)-representation is that new conditions can only be constructed out of given conditions by Boolean operations. As a consequence, it is, for example, not possible to define within a \( cis \) the condition ‘to be the grandfather of’ in terms of the conditions ‘to be the father of’ and ‘to be the mother of’. Note that if we want ‘grandfather’ to be a condition in our \( cis \) we can of course include it as a primitive condition.

With reference to the limitations mentioned above, it might be held that the \( cis \)-representation is too simple to be suitable for an overall representation of an actual legal system or a complex \( msic \)-systems of some other kind. Nevertheless, as will be apparent from the papers, the \( cis \)-representation is sufficiently rich to permit a detailed study of a number of issues pertaining especially to intermediate concepts in a legal system.\(^{15}\) The \( cis \)-representation can in a sense be viewed as an “idealized model” for studying different phenomena in \( msic \)-systems. When judging the usefulness of the \( cis \)-representation it is worth

\(^{15}\text{Cf. Lindahl \\& Odelstad (2006b), where it is suggested that a representation based on cylindric algebras would be more appropriate than a representation based on Boolean algebras.}\)
noting the following: Even if there are a number of difficulties when it comes
to a detailed representation of norms as joinings in a Boolean joining system,
it may be the case that these difficulties do not appear when the objective in
view is rather to construct an artificial normative system regulating an artificial
multiagent-system.

Condition implication structures are not the only kind of models of Boolean
joining systems that are interesting as representations of msic-systems. It is easy
to see that we can construct a Bjo out of a first order theory $\Sigma$. Consider the
structure $\langle B, \land', R \rangle$ where $\langle B, \land' \rangle$ is the Lindenbaum algebra of the predicate
calculus. Let $R$ be the quasi-ordering on $B$ determined by the Lindenbaum
algebra of $\Sigma$. Then $\langle B, \land', R \rangle$ is a Boolean quasi-ordering.

Boolean joining systems are obviously based on the notion of a Boolean
algebra. However, it is possible to define an analogous kind of systems based
on lattices. Such a system $\langle L_1, L_2, J \rangle$ consists of the lattice quasi-orderings
$L_1 = \langle L_1, \land, \lor, R_1 \rangle$ and $L_2 = \langle L_2, \land, \lor, R_2 \rangle$ and the set $J$ of joinings between
them and can be called a lattice joining system, abbreviated Ljs. A large
fraction of the formal result proved for Bjs’s will hold also for Ljs’s, roughly
because the complement operation in the Boolean algebras does not play a role
in the proofs. There are models of the theory of Ljs that can be interesting
representations of msic-systems. This holds, for example, when the concepts in
the msic-systems are not conditions but instead for instance aspects or equality
relations for aspects.

1.8 Applications in computer science

1.8.1 Introduction

The applications of the theory of msic-systems in computer science can follow
different paths. One path goes through the representation of normative systems
as msic-systems and the applications of normative systems in computer science.
Along a related path, the focus is on intermediate concepts, which are important
in normative systems but also in other kinds of systems, for example in knowl-
dge representation systems. A third path is the use of conceptual structures in
fields like the Semantic Web and information extraction. Since this thesis
focuses on normative systems, some comments on the role of normative systems
in computer science will be made.

1.8.2 Normative systems and computer science

There is a discipline emerging on the border between the formal study of norma-
tive systems and computer science. This discipline has, at least, a twofold
aim: on the one hand a computational approach to normative systems (primar-
ily regarding the law) and on the other hand the study of normative systems
used within computer science. The formal study of normative systems has been
one of the main objectives for that part of philosophical logic which is called
deontic logic dealing with the logic of obligation and permission. One of the main threads for presentations of the work within this discipline on the border between the formal study of normative systems and computer science is a series of biennial workshops on deontic logic in computer science, abbreviated ΔEON. The first ΔEON workshop took place in Amsterdam in 1991. The editors of the proceedings of the fifth ΔEON in 2000 describe the aim of the workshop as follows:

The present workshop, like its predecessors, was organized in order to bring together investigators working in deontic logic and its application in different areas of research: logic and philosophy, legal theory, computer science and artificial intelligence, management science, and other fields. During its brief history as an area of systematic logical research, deontic logic has developed from a study of conceptual and logical questions concerning the basic normative concepts—the concepts of obligation, permission and prohibition—into a complex interdisciplinary field which includes studies of norm systems and their formalisation, normative positions and the dynamics of norms, the representation of actions and agency, and reasoning about conflicting regulations, as well as research into reasoning about confidentiality and database security, the interaction between computer systems and their users, and the formalization of contracts and trade procedures. (Demolombe & Hilpinen, 2000, p. v.)

In the preface to the proceedings of the sixth ΔEON workshop held in London 2002, the editors stress that the workshops promote research and cooperation in a rapidly expanding interdisciplinary area, linking the formal-logical study of normative concepts and normative systems with computer science, artificial intelligence, organisation theory and law. In addition, there is currently a growing interest in this field of researchers in multi-agent systems and autonomous agents. (Horty & Jones, 2002, p. i.)

In Sergot (2001), the author refers to application of the theory of normative positions in the following fields:

- the formal representation of laws and legal contracts
- the specification of aspects of computer systems in the formal theory of organizations
- the analysis of notions such as responsibility, authorization and delegation
- agent-oriented programming
- agent communication languages.

Moreover, Sergot has constructed a computer program, Norman-G, based on the theory of normative positions. He describes the purpose of the program in the following way:
1.8. APPLICATIONS IN COMPUTER SCIENCE

The procedures described in previous sections have been implemented in a computer program, Norman-G, a prototype system intended to facilitate application of the theory to the analysis of practical examples, either for the purpose of interpretation and disambiguation of legal texts, rules, and regulations, or in the design and specification of a new set of norms. A typical example is the case discussed in [Jones & Sergot, 1992, 1993] concerning access ‘rights’ to sensitive medical information in a hospital database .... The problem here is to clarify and expand an incomplete and very imprecise statement of requirements into a precise specification at some desired level of detail. (Sergot, 2001, section 6.)

In the proceedings of the ninth ΔEON workshop in Luxemburg 2008 the editors state a number of possible areas for applications in computer science of the analysis based on deontic and action logic, among others the following (see van der Meyden & van der Torre, 2008, p.V):

- formal representation of legal knowledge and normative multiagent systems
- specification of systems for the management of bureaucratic processes in public and private administration
- specification of database integrity constraints computer security protocols
- analysis of deontic notions in the area of security and trust
- digital rights management and electronic contracts
- access control, security and privacy policies

The work presented in this thesis is intended as a contribution to the ‘ΔEON-discipline’. In papers (1), (3), (4) and (5) some legal examples are discussed as illustrations and applications of the theory developed. In paper (3), the legal exemplification plays a special role since the paper takes the form of comments on a legal example concerning acquisition of movable property by extinction of another person’s previous right. In paper (2), the possibility of using normative systems represented as npcis for regulating the behavior of multiagent-systems is investigated. Some aspects of this investigation will be outlined in the next section.

1.8.3 Agent oeconomicus norma

Within economic theory the consumer’s behaviour has traditionally been described as determined by a utility function. During the last three decades there has been a growing interest among researchers in how norms (for example rules of law) pose restrictions on the behaviour induced by the utility function. The behaviour of the consumers or other economic agents, according to this model, is the result of the interplay between optimization of the utility function and restrictions due to norms. We may perhaps speak of norm-regulated Homo oeconomicus. It has also been suggested that a model of this kind could be used for regulating the behaviour of artificial agents. We can perhaps call this model
Agent *oeconomicus norma*. The role that norms will have in regulating the behavior of agents is, according to this model, to delimit the autonomy of the agents. Metaphorically one can say that the norms define the scope (Spielraum) for an agent. The agent chooses the act it likes best within the scope determined by the norms.

Norm-regulation of agents presupposes a precise and significant representation of norms and normative systems. As was explained in previous sections, a norm is here represented as an implicative sentence where the antecedent is a descriptive condition stating the circumstances of an agent, and the consequent is a condition expressing the normative or deontic position that the agent has with respect to a state of affairs. Hence, from the norms of the system will follow a deontic structure over possible state of affairs implying that some states may be permissible while the rest are non-permissible. The “wish” or “desire” of an agent is represented as a preference structure over possible states or situations. The agent chooses an act which leads to one of the permissible states that it prefers the most.

In paper (2) the ideas outlined above were developed using the typology of normative (deontic) positions developed by Kanger and Lindahl and the algebraic representation of normative systems that Lindahl and I developed in paper (1) as well as earlier published papers. The aim of paper (2) was to present a model of how norms can be used to regulate the behaviour of multiagent-systems on the assumption that the role of norms is to define the Spielraum for an agent. An abstract architecture was defined in terms of a set-theoretical predicates and a MAS (a multiagent-system) having this architecture is called a norm-regulated DALMAS. One of the results in paper (2) was a scheme for how normative positions will restrict the set of actions that the agents are permitted to choose from.

### 1.8.4 Normative positions regulating actions

A DALMAS is an ordered 7-tuple \((\Omega, S, A, A, \Delta, \Pi, \Gamma)\) containing

- an agent set \(\Omega \{\omega, \kappa, \omega_1, \ldots \text{ elements in } \Omega\}\),

- a state or phase space \(S \{r, s, s_1, \ldots \text{ elements in } S\}\),

- an action set \(A\) such that for all \(a \in A, a : \Omega \times S \to S\) such that \(a(\omega, r) = s\) means that if the agent \(\omega\) performs the act \(a\) in state \(r\), then the result will be state \(s\) \((a, b, a_1, \ldots \text{ elements in } A)\),

- a function \(A : \Omega \times S \to \mathcal{P}(A)\) where \(\mathcal{P}(A)\) is the power set of \(A\); \(A(\omega, s)\) is the set of acts accessible (feasible) for agent \(\omega\) in state \(s\),

---

16 For the use of the term 'Spielraum' in this context, see Lindahl (1977) and Lindahl (2005).
17 The term DALMAS is chosen since the architecture is constructed for the application of deontic-action logic.
1.8. APPLICATIONS IN COMPUTER SCIENCE

- a deontic structure-operator $\Delta : \Omega \times S \to \mathcal{D}$ where $\mathcal{D}$ is a set of deontic structures of the same type with subsets of $A$ as domains and $\Delta(\omega, s)$ is $\omega$'s deontic structure on $A(\omega, s)$ in state $s$,

- a preference structure-operator $\Pi : \Omega \times S \to \mathcal{P}$ where $\mathcal{P}$ is a set of preference structures of the same type with subsets of $A$ as domains and $\Pi(\omega, s)$ is $\omega$'s preference structure on $A(\omega, s)$ in state $s$,

- a choice-set function $\Gamma : \Omega \times S \to \wp(A)$ where $\Gamma(\omega, s)$ is the set of actions for $\omega$ to choose from in state $s$.

Note that in the definition the Cartesian product $\Omega \times S$ motivates the introduction of a name for the elements in $\Omega \times S$. Let $\mathcal{D}$ be a DALMAS. A situation for the system $\mathcal{D}$ is determined by the agent to move $\omega$ and the state $s$. A situation is represented by an ordered pair $(\omega, s)$. The set of situations for $\mathcal{D}$ is thus $\Omega \times S$.

The idea behind a norm-regulated DALMAS is roughly the following: What is permissible for an agent to do in a situation $(\omega, s)$ is determined by a normative system $\mathcal{N}$. This idea can be explicated in the following way. Let

$$ T_d(\omega_1, \ldots, \omega_v, \omega; \omega, s) $$

mean that in the situation where it is the agent $\omega$'s turn to draw and the state of the system is $s$, $\omega$ has the normative position of type $T_d$ with regard to the state of affairs $d(\omega_1, \ldots, \omega_v)$.

Prohibited$_{\omega,s}(a)$ means that in the situation where it is $\omega$'s turn to draw and the state of the system is $s$, $\omega$ is prohibited to execute the act $a$.

The following seven principles establish connections between the condition $T_d$ and the predicate Prohibited (see paper (2) p. 160f.):

1. From $T_d(\omega_1, \ldots, \omega_v, \omega; \omega, s)$ follows no restriction on the acts.

2. From $T_d(\omega_1, \ldots, \omega_v, \omega; \omega, s)$ follows that if $d(\omega_1, \ldots, \omega_v; s)$ and $\neg d(\omega_1, \ldots, \omega_v; a(\omega, s))$ then Prohibited$_{\omega,s}(a)$.

3. From $T_d(\omega_1, \ldots, \omega_v, \omega; \omega, s)$ follows that if $[d(\omega_1, \ldots, \omega_v; s) \land d(\omega_1, \ldots, \omega_v; a(\omega, s))]$ then Prohibited$_{\omega,s}(a)$.

4. From $T_d(\omega_1, \ldots, \omega_v, \omega; \omega, s)$ follows that if $\neg d(\omega_1, \ldots, \omega_v; s)$ and $d(\omega_1, \ldots, \omega_v; a(\omega, s))$ then Prohibited$_{\omega,s}(a)$.

5. From $T_d(\omega_1, \ldots, \omega_v, \omega; \omega, s)$ follows that if $\neg d(\omega_1, \ldots, \omega_v; a(\omega, s))$ then Prohibited$_{\omega,s}(a)$.

6. From $T_d(\omega_1, \ldots, \omega_v, \omega; \omega, s)$ follows that if not $d(\omega_1, \ldots, \omega_v; s)$ if $d(\omega_1, \ldots, \omega_v; a(\omega, s))$ then Prohibited$_{\omega,s}(a)$.

7. From $T_d(\omega_1, \ldots, \omega_v, \omega; \omega, s)$ follows that if $d(\omega_1, \ldots, \omega_v; a(\omega, s))$ then Prohibited$_{\omega,s}(a)$. 

These principles can be used to define a deontic structure-operator $\Delta$ such that to each agent $\omega$ in a state $s$ is assigned the set of feasible acts $a$ that are not eliminated as $\text{Prohibited}_{\omega,s}(a)$ according to the rules (1)-(7) above. Since $\text{Prohibited}_{\omega,s}(a)$ is equivalent to $\neg \text{Permissible}_{\omega,s}(a)$, it follows that

$$\Delta(\omega, s) = \{ \text{Permissible}_{\omega,s}(a) : a \in A \}.$$ 

Note that at the outset, all feasible acts are permissible. The basic idea is that we eliminate elements from the set of permissible acts for $\omega$ in $s$ using the norms and sentences expressing what holds for the agents with respect to grounds in the norms.

The method used for representing norms in an architecture for norm-regulated MAS can be of importance for the effectiveness of the architecture. Here a few examples of what can be regarded as desiderata for a norm-representation method are mentioned.

1. The system of norms is depicted in a lucid, concise and effective way.
2. Changes and extensions of the normative system are easily described.
3. The normative system can be divided in different parts which can be changed independently.
4. The multi-agent system can by itself change the normative system wholly or partially.

The last item in the list may deserve a comment. It is often difficult to predict the effect of a normative system for a MAS or the effect of a change of norms. It is therefore desirable that the MAS can by itself evaluate the effect of the normative system and compare the result with other normative systems that it changes to. The result can be a kind of evolution of normative systems obtained by machine learning.

In paper (2), the $npcis$-model was used for representing normative systems, which resulted in an opportunity to test some aspects of this kind of representation in the area of multiagent-systems.

### 1.8.5 Prolog implementation of norm-regulated DALMAS

In Hjelmblom (2008), an implementation in Prolog of the theory of a norm-regulated DALMAS is presented. The algebraic theory is instrumentalyzed through an executable logic program. Important issues in the transition from a set-theoretical description to a Prolog implementation are discussed. Results include a general-level Prolog implementation, which may be freely used to implement specific systems.

The Prolog implementation gives a procedural semantics to the algebraic theory, see Lloyd (1987). Running the Prolog program has not only pedagogical value, but can aid understanding of the implications of changing parts of
1.9. NORMS AND FOREST CLEANING

the underlying theory. The fact that the Prolog program runs without notably long response time also testifies, albeit informally, to the acceptable computational complexity of the canonical model. Any domain-specific model created with Hjelmblom’s Prolog implementation can have its computational complexity analysed more formally through algorithmic analysis if necessary (see Purdom Jr. & Brown, 1985).

1.9 Norms and forest cleaning

1.9.1 Introduction

Forest management treatments presuppose, in a state of incomplete information, principles for choosing those trees that ought to be taken away and those that shall be left standing. In this section, which is a report on a work in progress carried out in cooperation with Ulla Ahonen-Jonnarth, the question is raised whether those principles can be structured as a combination of a normative system and a utility function. Of special interest is the possibility to evaluate the efficiency of the normative system and the utility function and, furthermore, suggest improvements of them. This section is based on Ahonen-Jonnarth & Odelstad (2005), Ahonen-Jonnarth & Odelstad (2006) and Odelstad (2007).

1.9.2 Cleaning in silico

In the forest industry there is an increasing interest in the automation of forest management treatments, perhaps with the ultimate goal that autonomous robots will be able to do a substantial part of such work. But before robots of this kind can be constructed many difficult problems must be solved, for example how the robots will perceive the environment and how they will transport themselves. But there are also decision-making problems involved. Three important kinds of forest management treatments are cleaning, thinning and harvesting, and they all require methods or principles for making decisions about which trees shall be removed and which will be left standing. Such “remove-decisions” must be made on-line with information based only on the robot’s nearest vicinity and about that part of the stand already cleared. The treatment cannot be evaluated until the actual stand is completely cleared. Testing and evaluating principles for remove-decisions by field experiments is expensive and time-consuming. It is therefore an interesting question whether evaluating experiments could be made in silico, i.e. through simulation.

In Ahonen-Jonnarth & Odelstad (2005), a platform for simulation of young forest stands is presented. Given field data of a special type of young forest, for example a 10-year-old, somewhat damp, spruce forest at 200 meters above sea level in the middle of Sweden, it is possible to simulate different stands of this type of forest. Field data of a few different types of young forests has so far been used for simulation. As a base for the simulation of different stands of the same forest type, it is of course also possible to use man-made, artificial data,
or to assign values to the parameters that govern the simulation.

One of the goals of our present work on automation of forest cleaning is to formulate different principles for making the remove-decisions, test the principles in simulated forests of different types and evaluate and compare the results. We are especially interested in the possibility that, given a method for evaluating the result of cleaning, the system can improve the decision-making principles and even suggest new ones on the basis of machine learning. How the principles for the remove-decisions ought to be formally represented seems to be a complicated question. One possibility we want to investigate is to use norm-regulated DALMAS as the architecture for a cleaning agent. At this preliminary stage, a cleaning agent is regarded as “a solitary being” and, hence, a cleaning Dalmas is a one-agent-system (thus more correctly a DaLoas), but we will here regard a one-agent-system as a degenerated MAS. But at a later stage, more then one agent may be involved, for example can ‘nature’ be regarded as an agent or can individual trees be regarded as agents. The last mentioned alternative is especially interesting if the growth of a forest stand is incorporated in the simulation.

A Dalmas can achieve the cleaning-decisions for a stand \( p \) in the following way. The stand is divided into \( n \) different areas. A state for the system is the stand with \( i \) areas cleaned, where \( 1 \leq i \leq n \), and a specification of what area to clean next. The initial state is the stand with 0 areas cleaned and the final state is the state with \( n \) areas cleaned. Let each area be denoted by a unique number between 1 and \( n \), and let \( S_i \) be the \( i \)th state. \( C_i \) denotes the set of cleaned areas and \( U_i \) the set of uncleaned areas in \( S_i \). Thus, \( C_i \cup U_i = \{1, 2, \ldots, n\} \) and \( C_i \cap U_i = \emptyset \). \( C_i \) contains \( i \) numbers and \( U_i \) contains \( n - i \) numbers. \( S_i = (C_i, U_i, j) \) where \( j \) is the area which will be cleaned next, i.e. \( j \in U_i \) and \( S_{i+1} = (C_i \cup \{j\}, U_i \setminus \{j\}, k) \) for some \( k \in U_i \setminus \{j\} \).

A few examples of possible norms regulating a cleaning Dalmas are given below:

(a) If there is only one undamaged tree in the area to be cleaned with a diameter within the desirable range, then this tree shall be saved.

(b) If there is at least one undamaged tree in the area to be cleaned with a diameter within the desirable range, then a damaged tree with a diameter below the desirable range may be taken away.

(c) If, in the area to be cleaned, a tree \( t \) is damaged and is closer than 0.5m to an undamaged tree with a diameter within the desirable range and with distances to other undamaged trees larger than 0.5m, then \( t \) may not be saved.

In many situations, the norms of a Dalmas do not determine the action to be taken in each state, but utility considerations are also necessary. Given a utility function we can search for the optimal way of cleaning the actual area, on the assumption that the cleaning satisfies the given norms.
For the possibility of using norms in the automation of forest cleaning in the way outlined above, it may be an important issue whether the cleaning system can optimize the system of norms regulating its remove-decisions. This is a special case of a more general problem: Suppose that $\mathcal{D}$ is a DALMAS, where the agents cooperate to solve a problem. Which normative system will lead to the most effective behavior of the system? It is desirable that $\mathcal{D}$ itself could determine the optimal normative system for the task in question. Given a set of grounds and a set of consequences, which together constitute the vocabulary of the system, $\mathcal{D}$ can test all possible sets of minimal norms (in many cases satisfying certain constraints, for example represented by intermediaries). If there is a function for evaluating the result of a run of $\mathcal{D}$, then different normative systems can be compared and the best system can be chosen. A change of vocabulary corresponds to a “mutation” among normative systems and can lead to dramatic changes in the effectiveness. Note that, in principle, the evaluation function can be very complicated, for example it can be multi-dimensional.
Chapter 2

Comments on the papers

2.1 The formal representation of msic-systems

The formal theory of msic-systems is to a large extent a question of representation. The algebraic framework for the representation of msic-systems in the work that Lindahl and I have conducted have gone through different “stages” and I will outline and discuss these stages here.

Stage 1: Lattice-representation

In Lindahl & Odelstad (1996) and (1999a), an msic-system is represented as a lattice $(L, \leq)$ of conditions extended with a quasi-ordering $\rho$. The lattice operations represent conjunction and disjunction respectively. Negation is not included for purely pragmatic reasons; in the first version of the theory we preferred to simplify the matter but still be able to express our main ideas about intermediaries. The partial ordering $\leq$ in the lattice represents “logical implication” and the quasi-ordering $\rho$ represent implications in a more general sense. The relation between the partial ordering $\leq$ and the quasi-ordering $\rho$ is such that the partial ordering $\leq_\rho$ generated from $\rho$ by the formation of equivalence classes is a lattice and $\leq$ is a subset of $\leq_\rho$. $(L_\rho, \leq_\rho)$ is the quotient algebra of $(L_\rho, \leq_\rho)$ with the respect to the indifference part of $\rho$. A two-sorted conceptual system is represented as a system of two sublattices $(L_1, \leq_1)$ and $(L_2, \leq_2)$ of $(L, \leq)$ and the set $\{ \langle x_1, x_2 \rangle \in L_1 \times L_2 \mid x_1 \leq_\rho x_2 \}$ of joinings between the sublattices.

Stage 2: Bqo-representation

In Odelstad & Lindahl (1998), the formal framework for representing msic-systems is modified in some respects:

1. We incorporate the operation of negation and suppose that the conditions constitute a Boolean algebra $\langle B, \wedge, ' \rangle$.

35
(2) We do not make a transition to the quotient algebra of \( \langle B, \land, \lor \rangle \) with respect to the indifference part of \( \rho \). Instead we construct the Boolean quasi-ordering \( \langle B, \land, \lor, \rho \rangle \). The reason is that we want to distinguish between two conditions even if they are indifferent with respect to \( \rho \). See paper (1), section 2.1.

(3) We make a clearer separation between the algebraic theories and the models used for the representation of \textit{msic}-systems. In stage 1, we regarded the lattice operations \( \land \) and \( \lor \) as representing conjunction and disjunction of conditions, since we only had one intended model in view. A \textit{Bqo} of conditions is one kind of \textit{Bqo}-model which can be used for representing \textit{msic}-systems and we do not exclude the possibility that there can be other kinds of models.

Note that an \textit{msic-system} is represented as a system of substructures of \( \langle B, \land, \lor, \rho \rangle \), called fragments, and the set of joinings between them. The formal tools for the representation of \textit{msic}-systems based on the \textit{Bqo}-theory is further developed in Odelstad & Lindahl (2000), Lindahl and Odelstad (2000) and paper (1) of this thesis. This \textit{Bqo}-representation is used in paper (2) and (3).

Stage 3: \textit{Bjs}-representation

In the \textit{Bqo}-representation of \textit{msic}-systems, strata of concepts of different sorts are represented as fragments of the basic \textit{Bqo} \( \langle B, \land, \lor, \rho \rangle \). Hence, \( B \) contains conditions of different sorts. But \( B \) contains also Boolean combinations of concepts of different sorts, i.e. compound concepts of a “mixed sort”. In many contexts, however, concepts of such mixed sorts are not of any interest and make the situation unnecessarily complicated. To avoid this complication, a \textit{Bjs} can be a useful tool for representations. A two-sorted conceptual system is then represented as a \textit{Bjs} \( \langle B_1, B_2, J \rangle \) consisting of two \textit{Bqo’s} \( B_1 \) and \( B_2 \) together with the set of joinings \( J \) between them. The \textit{Bqo’s} \( B_1 \) and \( B_2 \) are not necessarily fragments of one \textit{Bqo}. The axioms of a \textit{Bjs} are such that two fragments of a \textit{Bqo} and the joinings between them constitute a \textit{Bjs}.

However, if one wants to study \textit{msic}-systems containing conditions of several different sorts, this would involve a number of \textit{Bjs}’s related to each other in a complicated way. It may then be useful to have as a background a Boolean algebra \( \langle B, \land, \lor \rangle \) representing the “language” of the \textit{msic}-system and a binary relation \( \rho \) representing the non-logical (for example normative) content of the system. The sets of joinings between different strata of concepts will then be contained in \( \rho \). A \textit{msic-system} may therefore appropriately be represented as a \textit{supplemented Boolean algebra}, abbreviated \textit{sBa}, \( \langle B, \land, \lor, \rho \rangle \) with \textit{Bjs}’s lying within it. This is the approach in paper (4) and (5).

2.2 Related work

2.2.1 Intermediate concepts

The Scandinavian discussion of intermediate concepts (referred to in chapter 1, and in several of the papers) has had a crucial influence on the theory of \textit{msic}-systems put forth in this thesis. The following works have been of special
2.2. RELATED WORK

significance: Wedberg (1951), Ross (1951), Halldén (1978) and Lindahl (1985). Hedenius (1941) does not consider intermediate concepts but Hedenius' discussion about spurious and genuine norms is of great interest in this context. The works on introduction and elimination rules in logic and philosophy of mathematics by Gentzen, Dummett and Prawitz have, as emphasized above, also influenced this work. (See Gentzen 1934, Dummett 1973, and Prawitz 1977).

There are similarities between Richard Hare's prescriptiveism and the view of intermediaries developed in the work that Lindahl and I have conducted. In Lindahl & Odelstad (1999a), there is a reference to Hare (1989), but the relation between open intermediate concepts and prescriptiveism ought to be investigated in more detail.

What is not obvious from chapter 1 or the papers (but mentioned in Introduction) is the role of Bridgman's operationalism. The development of my thoughts about meaning has Bridgman's ideas about the subject as one of its starting-points. Extremely abbreviated, it can be stated that the role of operationalism in this context is the following: If a predicative concept is neither purely normative nor operationally definable, consider if it is an intermediate concept. To develop this dictum in detail is not, however, within the scope of this thesis.

2.2.2 The representation of normative systems

As is pointed out in paper 1 and 3, Alchourrón & Bulygin (1971) has influenced the way normative systems are represented in this thesis. In an earlier paper we describe our relation to Alchourrón and Bulygin in the following way:

The approach in the present paper is similar to that of Alchourrón and Bulygin in important respects. We study normative systems essentially as deductive mechanisms yielding outputs for inputs. Also, we are interested in a rational reconstruction of normative systems, where the result is more logically elaborated than the original version. Finally, a central problem is that of finding a suitable formal framework for representing the logically elaborated version. (Lindahl & Odelstad, 1999b, p. 91.)

In a series of papers, Makinson and van der Torre have developed a highly interesting theory called input-output logic, see for example Makinson and van der Torre (2000) and (2003). One striking similarity between input-output logic and the theory of msic-systems is that norms are represented as ordered pairs. This observation raises the question if there are some deep similarities between input-output logic and msic-theory. However, let me first state some obvious differences between the two theories. While msic-systems are by definition at least two-sorted, this does not holds for input-output logic. A common feature of the study of msic-systems reported here is the implicative closeness between strata of different sorts in an msic-system. An analogous study does not seem to have been carried out for input-output logic. The strata of an msic-system of
conditions are Boolean structures (\(B_0\)’s to be more precise), but the strata of \(msic\)-systems of other kinds need not be Boolean structures; instead, they can for example be lattice-like structures. In input-output logic, the set of inputs constitute a Boolean algebra and the same holds for the set of outputs.

The following remark sheds some light on the relation between input-output logic and the theory of \(msic\)-systems. (Knowledge of input-output logic is presupposed.) Suppose that \(\langle B_1, B_2, J \rangle\) is a \(Bjs\) where \(B_1 = \langle B_1, \land', R_1 \rangle\) and \(B_2 = \langle B_2, \land', R_2 \rangle\). Makinson and van der Torre state a number of rules for the output operators they define. Translated to a \(Bjs\) these rules are as follows:

**Strengthening Input:** From \(\langle a_1, a_2 \rangle \in J\) to \(\langle b_1, a_2 \rangle \in J\) whenever \(b_1 R_1 a_2\).

Follows from condition (1) of a \(Bjs\).

**Conjoining Input:** From \(\langle a_1, a_2 \rangle \in J\) and \(\langle a_1, b_2 \rangle \in J\) to \(\langle a_1, a_2 \land b_2 \rangle \in J\).

Follows from condition (3) of a \(Bjs\).

**Weakening Output:** From \(\langle a_1, a_2 \rangle \in J\) to \(\langle a_1, b_2 \rangle \in J\) whenever \(a_2 R_2 b_2\).

Follows from condition (1) of a \(Bjs\).

**Disjoining Input:** From \(\langle a_1, a_2 \rangle \in J\) and \(\langle b_1, a_2 \rangle \in J\) to \(\langle a_1 \lor b_1, a_2 \rangle \in J\).

Follows from condition (2) of a \(Bjs\).

There are three conditions on a joining space in a Boolean joining system. The comparison with input-output logic above shows that it could be of interest to define weaker kinds of systems characterized by, for example, condition (1) and (3).

### 2.2.3 Norms and artificial agent-systems

The works by Boman and Verhagen (see for example Boman 1998, Boman 1999, Verhagen & Boman 1999, Verhagen 2000, Boman et al. 2000) have been crucial to my interest in norms for multiagent-systems. The study of norm-regulations of artificial multiagent-system has during the latest decade developed into an active sub-discipline of AI called Normative Multi-Agent Systems. See Boella et al. (2007) for an overview of the area.

### 2.3 Non-Boolean joining systems

In this section, two examples of joining systems consisting of concepts but not constituting \(Bjs\) will be outlined briefly.

#### 2.3.1 Joining systems of equality-relations

The papers in this thesis have focused on \(msic\)-systems where the concepts are conditions subject to the Boolean operations. But there are kinds of conditions that do not constitute Boolean algebras. One example is equality-relations. The term ‘equality-relation’ here refer to a relation of equality with respect to some aspect \(\alpha\), and it is presupposed in this context that an equality-relation is always
an equivalence-relation, i.e. a reflexive, transitive and symmetric relation. Let $A$ be a non-empty set and let $E(A)$ be the set of equivalence relations on $A$. Define the binary relation $\leq$ on $E(A)$ in the following way: For all $\varepsilon_1, \varepsilon_2 \in E(A)$

\[ \varepsilon_1 \leq \varepsilon_2 \iff x\varepsilon_1 y \implies x\varepsilon_2 y. \]

(2.1)

The reader should be reminded of the following well-known fact. $E(A) = \langle E(A), \leq \rangle$ is a complete lattice. Note that the negation $\varepsilon'$ of an equivalence relation $\varepsilon \in E(A)$ is not an equivalence relation, i.e. $\varepsilon' \notin E(A)$. Let $E_1 = \langle E_1, \leq_1 \rangle$ and $E_2 = \langle E_2, \leq_2 \rangle$ be disjoint complete sublattices of $E(A)$ and consider $\langle E_1, E_2, J \rangle$ where $J = \leq / (E_1(A) \times E_2(A))$. Given some general conditions $\langle E_1, E_2, J \rangle$ is a joining system. We have here an example of a joining system which consists of conditions but they do not constitute a Boolean algebra.

A Boolean quasi-ordering is a Boolean algebra extended with a quasi-ordering satisfying certain conditions. We can define an analogous structure based on a lattice instead of a Boolean algebra. Let $E(A)$ and $\leq$ be as above and let $\langle E(A), \wedge, \vee \rangle$ be the lattice $\langle E(A), \leq \rangle$ expressed in terms of operations instead of a partial ordering, i.e. $\varepsilon_1 \wedge \varepsilon_2 = \inf \{\varepsilon_1, \varepsilon_2\}$ and $\varepsilon_1 \vee \varepsilon_2 = \sup \{\varepsilon_1, \varepsilon_2\}$. Suppose that $R$ is a quasi-ordering on $E(A)$ such that

1. $aRb$ and $aRc$ implies $aR(b \wedge c)$.
2. $aRc$ and $bRc$ implies $(a \lor b) Rc$.
3. $(a \land b)Ra$.
4. $aR(a \lor b)$.

Then $\langle E(A), \wedge, \lor, R \rangle$ is called a latticed quasi-ordering. The transition to the quotient algebra of $\langle E(A), \wedge, \lor \rangle$ with respect to the indifference part of $R$ will result in a lattice. (Cf. Lindahl & Odelstad, 1999a, p. 171.) The msic-systems consisting of equality-relations can often be represented as latticed quasi-orderings, and this also holds for msic-systems consisting of aspects.

### 2.3.2 Joining systems of aspects

The five papers in this thesis have focused on msic-systems where the concepts are conditions. But there are other kinds of concepts, for example aspects, in many disciplines called attributes. As examples of aspects let me mention a few: area, temperature, age, loudness and archeological value. It is a common view of aspects that they can, in some way or another, be represented as relational structures. In Odelstad (1992), a theory of aspects, where aspects are represented by systems of relationals, is set out. A relational is a function with sets as arguments and structures as values. On sets of systems of relationals, several quasi-orderings can be defined but here only one example will be given.

Let $\text{Rel}(D)$ denote the set of systems of relationals whose range of definition is the family $D$ of sets. This means that for all $R \in \text{Rel}(D)$ it holds that $R = \langle \rho_i \rangle_{i \in I}$ for some set $I$ and for all $A \in D$, $\rho_i(A) \subseteq A^\nu_i$ where $\nu_i$ is the arity of the relational $\rho_i$. Hence, $\text{Rel}(A) = \langle A, \rho_i \rangle_{i \in I}$. Let $\text{Sub}(\text{Rel}(A), \text{Rel}(A))$ denote the set of isomorphisms from $\text{Rel}(A)$ to $\text{Rel}(A)$. We can define a relation $\text{sub}$ on $\text{Rel}(D)$ in the following way: If $R_1, R_2 \in \text{Rel}(D)$ then
\[ R_2 \text{ sub } R_1 \text{ iff for all } A, B \in \mathcal{D} : \exists (R_2(A), R_2(B)) \supset \exists (R_1(A), R_1(B)). \tag{2.2} \]

It is obvious that \text{sub} is a quasi-ordering on Rel \mathcal{D}. It follows from Odelstad (1992) that \langle \text{Rel} \mathcal{D}, \text{sub} \rangle is a complete quasi-lattice and it is therefore possible that there are joining systems lying within \langle \text{Rel} \mathcal{D}, \text{sub} \rangle.\footnote{If \langle A, R \rangle is a quasi-ordering such that lub \_ \text{ of } a \_ b \_ \varnothing \text{ and glb } \_ \text{ of } a \_ b \_ \varnothing \text{ for all } a, b \_ A \_ a, then \langle A, R \rangle \_ \text{ will be called a quasi-lattice. If lub } \_ \text{ of } X \_ \varnothing \text{ and glb } \_ \text{ of } X \_ \varnothing \text{ for all } X \_ A \_ X, then a quasi-ordering \langle A, R \rangle \_ \text{ is a complete quasi-lattice. Suppose that \langle A, R \rangle \_ \text{ is a quasi-lattice, } Q \_ \text{ the equality-part of } R \_ \text{ and } A_Q \_ \text{ is the set of Q-} \text{equivalence classes generated by elements of } A \ _ \text{. Then } \langle A_Q, \rho \_ \text{, where } [a]_{Q} \_ \rho [b]_{Q} \_ \text{ iff } a \_ R b \_ \text{, is a lattice. If } \langle A, R \rangle \_ \text{ is a complete quasi-lattice then } \langle A_Q, \rho \_ \text{ is a complete lattice.}}}

The relational systems in \langle \text{Rel} \mathcal{D}, \text{sub} \rangle can be of different sorts and it is a meaningful question if they form joining systems or even latticed joining systems. Note that in \langle \text{Rel} \mathcal{D}, \text{sub} \rangle the implicational relation sub is not implication in the usual sense but expresses a kind of dependence relation.

### 2.4 Two comments on DALMAS

#### 2.4.1 The move-operator

The Move-operator defined in paper (2) p. 156 is not quite adequate, which is pointed out in Hjelmblom (2008), where also a correction is suggested:

Recent studies [Olsson, 2006] have discovered that the Move-operator as defined in [Odelstad & Boman (2004), paper (2)] does not give sufficient expressiveness to allow the formulation of an important class of norms. This issue has been discussed with the authors of the DALMAS architecture [Odelstad & Boman (2004)] who have suggested an extension of the abstract architecture: the \text{M}_n-operator. This operator is very similar to the original \text{M}_n-operator, but it also unifies the acting agent with one of the agent arguments of the sit-condition that the operator is applied to:

\[ M_n c(\omega_1, \ldots, \omega_{\nu}, \omega_{\nu+1}; s) \_ \text{iff } \omega_{\nu+1} = \omega \_ s \_ c(\omega_1, \ldots, \omega_{\nu}; s) \_ \omega_{\nu+1} = \omega_n \]

where \( \omega_n \) is the \( n \)-th agent argument of \( c \). (Hjelmblom, 2008, p. 15.)

#### 2.4.2 A note on Prohibited

In paper (2), the Kanger-Lindahl theory of normative positions is applied to the DALMAS setting. The predicate Prohibited which is central for the norm-regulation of a DALMAS is defined within the framework of the Kanger-Lindahl theory. The dynamics of a DALMAS is somewhat special. The transition from one state to another is in each case the result of one agent’s action, and given a state it is meaningful to talk about the previous state and the subsequent state.
2.4. TWO COMMENTS ON DALMAS

The application of Kangor-Lindahl theory is the basis for the definition of the predicate Prohibited, but the application in the DALMAS-context does not seem to be as straightforward as I thought when paper (2) was written.² I will here try to make clear the idea behind the definition of Prohibited and point out a possibly different way to approach the application of the Kangor-Lindahl theory in this context.

Let us consider the transition from state $s$ to the next state $s^+$ when $\omega$ is to move, and let us focus on the $\nu$-ary condition $d$. As a simplification, the sequence of agents $\omega_1, \ldots, \omega_\nu$ is omitted, and $d(\omega_1, \ldots, \omega_\nu; \omega, s)$ is abbreviated as $d(\omega, s)$ and $d(\omega_1, \ldots, \omega_\nu; s^+)$ is abbreviated as $d(s^+)$. With regard to $d$, there are four possible alternatives for the transition from $s$ to $s^+$, since in $s$ as well as in $s^+$, $d$ or $\neg d$ could hold:

(I) $d(\omega, s) \land d(s^+)$

(II) $\neg d(\omega, s) \land d(s^+)$

(III) $d(\omega, s) \land \neg d(s^+)$

(IV) $\neg d(\omega, s) \land \neg d(s^+)$

Note that the transitions (I) and (III) constitute a contradiction and the same holds of (II) and (IV).

The following statements seem non-controversial: If (II) is the case then $\omega$ sees to it that $d$ holds in $s^+$. If (III) is the case then $\omega$ sees to it that $\neg d$ holds in $s^+$. If (I) as well as (IV) is the case then $\omega$ is passive with respect to $d$.

However, the idea behind Prohibited is made clearer if we start from sentences expressing what “may not be seen to”. The following principles seem reasonable:

$\neg$MayDo($\omega, d(s^+)$) implies that (II) is an excluded alternative.

$\neg$MayDo($\omega, \neg d(s^+)$) implies that (III) is an excluded alternative.

$\neg$MayPass($\omega, \neg d(s^+)$) implies that (I) and (IV) taken together is an excluded alternative, i.e. both (I) and (IV) is excluded.

Given that an alternative that is excluded implies a prohibition, we arrive at the following interpretation of the type-operators $T_i$:

$T_1d(\omega, s)$, i.e. MayDo($\omega, d(s^+)$) & MayPass($\omega, d(s^+)$) & MayDo($\omega, \neg d(s^+)$)

None of (I)-(IV) is excluded.

$T_2d(\omega, s)$, i.e. MayDo($\omega, d(s^+)$) & MayPass($\omega, d(s^+)$) & $\neg$MayDo($\omega, \neg d(s^+)$)

(III) is excluded, confer $E_2$.

$T_3d(\omega, s)$, i.e. MayDo($\omega, d(s^+)$) & $\neg$MayPass($\omega, d(s^+)$) & MayDo($\omega, \neg d(s^+)$).

That both (I) and (IV) hold is excluded, confer $E_3$.

$T_3d(\omega, s)$, i.e. $\neg$MayDo($\omega, d(s^+)$) & MayPass($\omega, d(s^+)$) & MayDo($\omega, \neg d(s^+)$).

²I am grateful to Magnus Hjelmblom for pointing this out to me.
(III) is excluded, confer \( E_4 \).

\( T_3 d(\omega, s) \), i.e. \( \neg \text{MayDo}(\omega, d(s^+)) \& \neg \text{MayPass}(\omega, d(s^+)) \& \neg \text{MayDo}(\omega, \neg d(s^+)) \).

That both (I) and (IV) hold is excluded, and furthermore, (III) is excluded. Since (I) and (III) is impossible, (III) and (IV) are excluded, confer \( E_5 \).

\( T_4 d(\omega, s) \), i.e. \( \neg \text{MayDo}(\omega, d(s^+)) \& \text{MayPass}(\omega, d(s^+)) \& \neg \text{MayDo}(\omega, \neg d(s^+)) \).

(II) and (III) are excluded, confer \( E_6 \).

\( T_7 d(\omega, s) \), i.e. \( \neg \text{MayDo}(\omega, d(s^+)) \& \neg \text{MayPass}(\omega, d(s^+)) \& \text{MayDo}(\omega, \neg d(s^+)) \).

(II) is excluded, and that both (I) and (IV) hold is excluded. Since (II) and (IV) is impossible it follows that (II) and (I) are excluded, confer \( E_7 \).

We have above looked at what kind of transitions that are excluded given negations of sentences containing the combination of May and Do or Pass. Another way to look at Do and Pass is the following. For triples of the form \( \langle \omega, d(\omega_1, ..., \omega_n); s \rangle \) three operators Do, Pass and Act are defined in the following way (as before \( d(\omega_1, ..., \omega_n) \) is shortened as \( d \)).

\[
\text{Do}(\omega, d; s) = \begin{cases} 
  d(s^+) \text{ if } d(s) \text{ (cf. I)} \\
  d(s^+) \text{ if } \neg d(s) \text{ (cf. II)} 
\end{cases}
\]

\[
\text{Pass}(\omega, d; s) = \begin{cases} 
  d(s^+) \text{ if } d(s) \text{ (cf. I)} \\
  \neg d(s^+) \text{ if } \neg d(s) \text{ (cf. IV)} 
\end{cases}
\]

\[
\text{Act}(\omega, d; s) = \begin{cases} 
  \neg d(s^+) \text{ if } d(s) \text{ (cf. III)} \\
  d(s^+) \text{ if } \neg d(s) \text{ (cf. II)} 
\end{cases}
\]

‘\( \text{Do}(\omega, d; s) \)’ is intended to mean that \( \omega \) sees to it in state \( s \) that \( d \) is true in state \( s^+ \). The interpretation of ‘\( \text{Pass}(\omega, d; s) \)’ is that \( \omega \) is passive in state \( s \) with respect to \( d \), i.e. leaves \( d \) as it is. The interpretation of ‘\( \text{Act}(\omega, d; s) \)’ is that \( \omega \) is active in state \( s \) with respect to \( d \), i.e. always changes the truth value of \( d \) whatever it is.

Note that

\[
\text{Do}(\omega, \neg d; s) = \begin{cases} 
  \neg d(s^+) \text{ if } d(s) \text{ (cf. III)} \\
  \neg d(s^+) \text{ if } \neg d(s) \text{ (cf. IV)} 
\end{cases}
\]

and note also that

\[
\text{Pass}(\omega, d; s) = \text{Pass}(\omega, \neg d; s)
\]

and

\[
\text{Act}(\omega, d; s) = \text{Act}(\omega, \neg d; s).
\]

As observed above, the transitions (I) and (III) taken together constitute a contradiction and the same holds of (II) and (IV). The consistent pairs of (I)-(IV) correspond to \( \text{Do}(\omega, d; s), \text{Do}(\omega, \neg d; s), \text{Pass}(\omega, d; s) \) or \( \text{Act}(\omega, d; s) \). From this follows that not \( \text{Do}(\omega, d; s) \) and not \( \text{Do}(\omega, \neg d; s) \) implies \( \text{Pass}(\omega, d; s) \) or
2.4. TWO COMMENTS ON DALMAS

\(\text{Act}(\omega, d; s)\). In paper (2) p. 147 it is said that \(x\)'s passivity with regard to \(q\) is expressed by the formula

\[-\text{Do} (x, q) \& \neg \text{Do} (x, -q)\]  \hspace{1cm} (2.3)

and this is abbreviated as \(\text{Pass}(x, q)\). But this seems to disregard the possibility that \(x\) is with regard to \(q\) always active. What this will imply for the definition of Prohibited will here be left open.
CHAPTER 2. COMMENTS ON THE PAPERS
Part II

The Papers
The following papers are reprinted by kind permission of *Springer Science and Business Media*:


The following papers are reprinted by kind permission of *Elsevier*:


Appendix A

Proofs of the results in paper (5)

The proofs of the results in paper (5) were included in the version of the paper submitted to the conference ΔΕΟΝ’08 but were omitted in the final version due to lack of space. They are in revised form presented here instead. (Note that the numbering of the propositions, theorems and corollaries is not the same as in paper (5).)

A.0.3  Weakest grounds and strongest consequences

Proposition 5  (i) Suppose that \( \langle B_1, B_2, J \rangle \) is a Bjs. Suppose further that \( a_1 R_1 b_1, (b_1, b_2) \in J \) and \( b_2 R_2 a_2 \). Then \( \langle a_1, a_2 \rangle \in J \).

(ii) Suppose that \( \langle B_1, B_2, J \rangle \) is a Bjs, \( \text{WG} (a_1, a_2, B_1) \) and \( \text{WG} (b_1, b_2, B_1) \). If \( a_2 R_{2b2} \) then \( a_1 R_{1b1} \).

(iii) If \( \langle B_1, B_2, J \rangle \) is a Bjs, \( \text{WG} (a_1, a_2, B_1) \) and \( \text{WG} (b_1, a_2, B_1) \), then \( a_1 Q_{1b1} \).

(iv) Suppose that \( \langle B_1, B_2, J \rangle \) is a Bjs, \( \text{SC} (a_2, a_1, B_2) \) and \( \text{SC} (b_2, b_1, B_2) \). If \( a_1 R_{1b1} \) then \( a_2 R_{2b2} \).

(v) If \( \langle B_1, B_2, J \rangle \) is a Bjs, \( \text{SC} (a_2, a_1, B_2) \) and \( \text{SC} (b_2, a_1, B_2) \) then \( a_2 Q_{2b2} \).

(vi) If \( \text{WG} (a_1, a_2, B_1) \) and \( \text{WG} (b_1, b_2, B_1) \) then \( \text{WG} (a_1 \land b_1, a_2 \land b_2, B_1) \).

(vii) If \( \text{SC} (a_2, a_1, B_2) \) and \( \text{SC} (b_2, b_1, B_2) \) then \( \text{SC} (a_2 \lor b_2, a_1 \lor b_1, B_2) \).

Proof. (i) Immediate consequence of the definition of a Bjs.

(ii) \( a_1 J a_2 R_{2b2} \), hence \( a_1 J b_2 \), from which follows that \( a_1 R_{1b1} \) since \( \text{WG} (b_1, b_2, B_1) \).

(iii) Obvious.

(iv) \( a_1 R_{1b1} J b_2 \), hence \( a_1 J b_2 \), from which follows that \( a_2 R_{2b2} \) since \( \text{SC} (a_2, a_1, B_2) \).
(v) Obvious.

(vi) $a_1 \land b_1 J a_2 \land b_2$. Suppose that $c_1 J a_2 \land b_2$. Then $c_1 J a_2$ and $c_1 J b_2$, and hence $c_1 R_1 a_1$ and $c_1 R_1 b_1$ which implies that $c_1 R_1 a_1 \land b_1$. Thus $WG\langle a_1 \land b_1, a_2 \land b_2, B_1 \rangle$.

(vii) $a_1 \lor b_1 J a_2 \lor b_2$. Suppose that $a_1 \lor b_1 J c_2$. Then $a_1 J c_2$ and $b_1 J c_2$, and hence $a_2 R_2 a_2$ and $b_2 R_2 b_2$ which implies that $a_2 \lor b_2 R_2 c_2$. ■

A.0.4 Minimality

First two propositions that together prove the following theorem:

**Theorem 6** Suppose that $\langle B_1, B_2, J \rangle$ is a Bjs. Then $\langle a_1, a_2 \rangle \in \min J$ iff $WG\langle a_1, a_2, B_1 \rangle$ and $SC\langle a_2, a_1, B_2 \rangle$.

**Proposition 7** Suppose that $\langle B_1, B_2, J \rangle$ is a Bjs. Suppose further that $\langle a_1, a_2 \rangle \in \min J$ and that $WG\langle b_1, a_2, B_1 \rangle$ and $SC\langle b_2, a_1, B_2 \rangle$. Then $a_1 Q_1 b_1$ and $a_2 Q_2 b_2$.

**Proof.** (I) From $\langle a_1, a_2 \rangle \in J$ and $WG\langle b_1, a_2, B_1 \rangle$ follows $a_1 R_1 b_1$. Suppose now that $a_1 S_1 b_1$. Since $\langle b_1, a_2 \rangle \in J$ it follows that $\langle b_1, a_2 \rangle < \langle a_1, a_2 \rangle$ which contradicts the assumption that $\langle a_1, a_2 \rangle \in \min J$. Hence, $a_1 Q_1 b_1$.

(II) From $\langle a_1, a_2 \rangle \in J$ and $SC\langle b_2, a_1, B_2 \rangle$ it follows that $b_2 R_2 a_2$. Suppose $b_2 S_2 a_2$. Then $\langle a_1, b_2 \rangle < \langle a_1, a_2 \rangle$ which contradicts the assumption that $\langle a_1, a_2 \rangle \in \min J$. Hence, $a_2 Q_2 b_2$. ■

**Corollary 8** If $\langle B_1, B_2, J \rangle$ is a Bjs and $\langle a_1, a_2 \rangle \in \min J$, then $WG\langle a_1, a_2, B_1 \rangle$ and $SC\langle a_2, a_1, B_2 \rangle$.

**Proposition 9** Suppose that $\langle B_1, B_2, J \rangle$ is a Bjs. Suppose further that $WG\langle a_1, a_2, B_1 \rangle$ and $SC\langle a_2, a_1, B_2 \rangle$. Then $\langle a_1, a_2 \rangle \in \min J$.

**Proof.** From the assumption follows that $\langle a_1, a_2 \rangle \in J$. Suppose now that there is $\langle b_1, b_2 \rangle \in J$ such that $\langle b_1, b_2 \rangle < \langle a_1, a_2 \rangle$, from which follows that $a_1 R_1 b_1$ and $b_2 R_2 a_2$. $\langle b_1, b_2 \rangle \in J$ and $b_2 R_2 a_2$ implies that $\langle b_1, a_2 \rangle \in J$. Since $WG\langle a_1, a_2, B_1 \rangle$ it follows that $b_1 R_1 a_1$, $a_1 R_1 b_1$ and $\langle b_1, b_2 \rangle \in J$ implies that $\langle a_1, b_2 \rangle \in J$. Since $SC\langle a_2, a_1, B_2 \rangle$ it follows that $a_2 R_2 b_2$. Thus the following holds: $a_1 R_1 b_1$ and $b_1 R_1 a_1$, which implies $a_1 Q_1 b_1$, and $b_2 R_2 a_2$ and $a_2 R_2 b_2$, which implies $a_2 Q_2 b_2$. This contradicts the assumption that there is $\langle b_1, b_2 \rangle \in J$ such that $\langle b_1, b_2 \rangle < \langle a_1, a_2 \rangle$, which proves that $\langle a_1, a_2 \rangle \in \min J$. ■

The last proposition and the corollary can be summarized in theorem 6 above.

**Proposition 10** Suppose that $\langle B_1, B_2, J \rangle$ is a Bjs which satisfies connectivity. Suppose further that $WG\langle a_1, a_2, B_1 \rangle$. Then there is $b_2 \in B_2 : \langle a_1, b_2 \rangle \in \min J$ and $b_2 R_2 a_2$. 
Proof. From WG \((a_1, a_2, B_1)\) follows \(\langle a_1, a_2 \rangle \in J\). Connectivity implies that there is \(b_1 \in B_1\) and \(b_2 \in B_2\) such that \(\langle b_1, b_2 \rangle \in \min J\) and \(\langle b_1, b_2 \rangle \leq \langle a_1, a_2 \rangle\). Hence, \(a_1 R_1 b_1\) and \(b_2 R_2 a_2\). \(\langle b_1, b_2 \rangle \in J\) and \(b_2 R_2 a_2\) implies \(\langle b_1, a_2 \rangle \in J\). Since WG \((a_1, a_2, B_1)\) it follows that \(b_1 R_1 a_1\), which together with \(a_1 R_1 b_1\) implies \(a_1 Q_1 b_1\). Hence, \(\langle a_1, b_2 \rangle \in \min J\).

Proposition 11 Suppose that \(\langle B_1, B_2, J \rangle\) is a Bjs which satisfies connectivity. Suppose further that SC \((a_2, a_1, B_2)\). Then there is \(b_1 \in B_1 : \langle b_1, a_2 \rangle \in \min J\) and \(a_1 R_1 b_1\).

Proof. From SC \((a_2, a_1, B_2)\) follows \(\langle a_1, a_2 \rangle \in J\). Connectivity implies that there is \(b_1 \in B_1\) and \(b_2 \in B_2\) such that \(\langle b_1, b_2 \rangle \in \min J\) and \(\langle b_1, b_2 \rangle \leq \langle a_1, a_2 \rangle\). Hence, \(a_1 R_1 b_1\) and \(b_2 R_2 a_2\). \(\langle b_1, b_2 \rangle \in J\) and \(a_1 R_1 b_1\) implies \(\langle a_1, b_2 \rangle \in J\). Since SC \((a_2, a_1, B_2)\) it follows that \(a_2 R_2 b_2\), which together with \(b_2 R_2 a_2\) implies \(a_2 Q_2 b_2\). Hence, \(\langle b_1, a_2 \rangle \in \min J\).

Proposition 12 Suppose that \(\langle B_1, B_2, J \rangle\) is a Bjs that satisfies connectivity and \(\langle a_1, a_2 \rangle \in \min J\). If \(\langle a_1, b_2 \rangle \in J\) then \(a_2 R_2 b_2\) and if \(\langle b_1, a_2 \rangle \in J\) then \(b_1 R_1 a_1\).

Proof. (1) Suppose that \(\langle a_1, a_2 \rangle \in \min J\) and \(\langle a_1, b_2 \rangle \in J\). Since J satisfies connectivity there is \(\langle c_1, c_2 \rangle \in \min J\) such that \(\langle c_1, c_2 \rangle \leq \langle a_1, b_2 \rangle\). Hence, \(a_1 R_1 c_1\) and \(b_2 R_2 b_2\). From theorem 6 follows that SC \(\langle a_2, a_1, B_2\rangle\) and SC \(\langle c_2, c_1, B_2\rangle\). Since \(a_1 R_1 c_1\) it follows according to proposition 5 (iv) that \(a_2 R_2 c_2\). Together with \(c_2 R_2 b_2\) this implies \(a_2 R_2 b_2\).

(2) Suppose that \(\langle a_1, a_2 \rangle \in \min J\) and \(\langle b_1, a_2 \rangle \in J\). Since J satisfies connectivity there is \(\langle d_1, d_2 \rangle \in \min J\) such that \(\langle d_1, d_2 \rangle \leq \langle b_1, a_2 \rangle\). Hence, \(b_1 R_1 d_1\) and \(d_2 R_2 a_2\). From theorem 6 follows that WG \(\langle a_1, a_2, B_1\rangle\) and WG \(\langle d_1, d_2, B_1\rangle\). Since \(d_2 R_2 a_2\) it follows according to proposition 5 (iii) that \(d_1 R_1 a_1\). Together with \(b_1 R_1 d_1\) this gives \(b_1 R_1 a_1\).

Corollary 13 (i) Suppose that \(\langle B_1, B_2, J \rangle\) is a Bjs that satisfies connectivity. If \(\langle a_1, a_2 \rangle, (b_1, b_2) \in \min J\) then \(a_1 R_1 b_1\) iff \(a_2 R_2 b_2\).

(ii) Suppose that \(\langle B_1, B_2, J \rangle\) is a Bjs that satisfies connectivity and \(\langle a_1, a_2 \rangle \in \min J\). If \(\langle a_1, b_2 \rangle \in \min J\) then \(a_2 R_2 b_2\) and if \(\langle b_1, a_2 \rangle \in \min J\) then \(a_1 Q_1 b_1\).

Proof. (i) Suppose \(a_1 R_1 b_1\). Then \(\langle a_1, b_2 \rangle \in J\) and since \(\langle a_1, a_2 \rangle \in \min J\) it follows that \(a_2 R_2 b_2\). Suppose that \(a_2 R_2 b_2\). Then \(\langle a_1, b_2 \rangle \in J\) and since \(\langle b_1, b_2 \rangle \in \min J\) it follows that \(a_1 R_1 b_1\).

(ii) is an immediate consequence of (i).

Theorem 14 Suppose that \(\langle B_1, B_2, J \rangle\) is a Bjs that satisfies connectivity. If \(\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in \min J\), then there is \(c_2 \in B_2\) and \(d_1 \in B_1\) such that \(\langle a_1 \wedge b_1, c_2 \rangle \in \min J\) and \(\langle d_1, a_2 \vee b_2 \rangle \in \min J\), and, furthermore, it holds that \(c_2 R_2 a_2 \wedge b_2\) and \(a_1 \vee b_1 R_1 d_1\).

Proof. (1) Since \(\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in J\) it follows that \(\langle a_1 \wedge b_1, a_2 \wedge b_2 \rangle \in J\). From connectivity follows that there is \(\langle c_1, c_2 \rangle \in \min J\) such that \(\langle c_1, c_2 \rangle \leq\)
\langle a_1 \land b_1, a_2 \land b_2 \rangle$, which implies that $a_1 \land b_1 \mathcal{R}_1 c_1$ and $c_2 \mathcal{R}_2 a_2 \land b_2$. From this follows that $c_3 \mathcal{R}_3 a_3$ and $c_2 \mathcal{R}_2 b_2$. Since $\langle a_1, a_2 \rangle, \langle c_1, c_2 \rangle \in \min J$ it follows from $c_3 \mathcal{R}_3 a_3$ according to corollary 13 that $c_1 \mathcal{R}_1 a_1$. Since $\langle b_1, b_2 \rangle, \langle c_1, c_2 \rangle \in \min J$ if follows from $c_2 \mathcal{R}_2 b_2$ according to corollary 13 that $c_1 \mathcal{R}_1 b_1$. $c_1 \mathcal{R}_1 a_1$ and $c_1 \mathcal{R}_1 b_1$ implies that $c_1 \mathcal{R}_1 a_1 \land b_1$ according to condition (1) in the definition of a B\textit{go}. Since it also holds that $a_1 \land b_1 \mathcal{R}_1 c_1$ it follows that $a_1 \land b_1 \mathcal{R}_1 c_1$. Hence, $\langle a_1 \land b_1, c_2 \rangle \in \min J$. Note that $c_2 \mathcal{R}_2 a_2 \land b_2$.

(2) Since $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in \min J$ it follows that $\langle a_1 \lor b_1, a_2 \lor b_2 \rangle \in \min J$. From connectivity follows that there is (d1, d2) \in \min J such that
\begin{align*}
\langle d_1, d_2 \rangle \subseteq \langle a_1 \lor b_1, a_2 \lor b_2 \rangle,
\end{align*}
which implies that $a_1 \lor b_1 \mathcal{R}_1 d_1$ and $d_2 \mathcal{R}_2 a_2 \lor b_2$. From this follows that $a_1 \mathcal{R}_1 d_1$ and $b_1 \mathcal{R}_1 d_1$. Since $\langle a_1, a_2 \rangle, \langle d_1, d_2 \rangle \in \min J$ if follows from $a_1 \mathcal{R}_1 d_1$ according to corollary 13 that $a_2 \mathcal{R}_2 d_2$. Since $\langle b_1, b_2 \rangle, \langle d_1, d_2 \rangle \in \min J$ if follows from $b_1 \mathcal{R}_1 d_1$ according to corollary 13 that $b_2 \mathcal{R}_2 d_2$. $a_2 \mathcal{R}_2 d_2$ and $b_2 \mathcal{R}_2 d_2$ implies that $a_2 \lor b_2 \mathcal{R}_2 d_2$. Since it also holds that $d_2 \mathcal{R}_2 a_2 \lor b_2$ it follows that $a_2 \lor b_2 \mathcal{R}_2 d_2$. Hence, $\langle d_1, a_2 \lor b_2 \rangle \in \min J$. Note that $a_1 \lor b_1 \mathcal{R}_1 d_1$.

A.0.5 Intervenients

In this section we suppose that $\mathcal{S} = (B, \land, \land', \rho)$ is an s\textit{B}a and that $\langle B_1, B_2, J_{1,2} \rangle, \langle B_2, B_3, J_{2,3} \rangle$ and $\langle B_1, B_3, J_{1,3} \rangle$ are B\textit{a}s lying within $\mathcal{S}$ and satisfying connectivity. Suppose further that $J_{1,3} = J_{1,2} \cap J_{2,3}$. This means that $\langle a_1, a_3 \rangle \in J_{1,3}$ iff there is $a_2 \in B_2$ such that $\langle a_1, a_2 \rangle \in J_{1,2}$ and $\langle a_2, a_3 \rangle \in J_{2,3}$.

Proposition 15 If $a_2 \mathcal{R}_2 \langle a_1, a_3 \rangle$ and $a_2 \mathcal{R}_2 \langle b_1, b_3 \rangle$ then $a_1 \mathcal{R}_1 b_1$ and $a_3 \mathcal{R}_3 b_3$.

Proof. WG (a1, a2, B1), SC (a3, a2, B3) and WG (b1, a2, B1), SC (b3, a2, B3). Hence, according to 5 (iii), $a_1 \mathcal{R}_1 b_1$ and, according to proposition 5 (v) $a_3 \mathcal{R}_3 b_3$.

Theorem 16 $\min J_{1,2} | \min J_{2,3} \subseteq \min J_{1,3}$.

Proof. Suppose that $\langle a_1, a_2 \rangle \in \min J_{1,2}$ and $\langle a_2, a_3 \rangle \in \min J_{2,3}$. Since $J_{1,3} = J_{1,2} \cap J_{2,3}$ it follows that $\langle a_1, a_3 \rangle \in J_{1,3}$. (B1, B2, J1,3) satisfies connectivity which implies that there is $b_1 \in B_1$ such that $\langle b_1, b_2 \rangle \subseteq \langle a_1, a_3 \rangle$, i.e. $a_1 \mathcal{R}_1 b_1$ and $b_3 \mathcal{R}_3 a_3$. Since $J_{1,3} = J_{1,2} \cap J_{2,3}$ it follows that there is $b_2 \in B_2$ such that $\langle b_1, b_2 \rangle \in J_{1,2}$ and $\langle b_2, b_3 \rangle \in J_{2,3}$. Hence, $\langle a_1, a_2 \rangle \in J_{1,2}$ and $\langle a_2, a_3 \rangle \in J_{2,3}$, which implies $a_2 \mathcal{R}_2 b_2$ and $b_2 \mathcal{R}_2 a_2$. According to corollary 13 (ii), from this follows, since $\langle a_2, a_3 \rangle \in \min J_{2,3}$, $\langle b_2, a_3 \rangle \in \min J_{2,3}$ and since $\langle b_2, b_3 \rangle \in J_{2,3}$ it follows that $a_3 \mathcal{R}_3 b_3$. But, as pointed out above, $b_3 \mathcal{R}_3 a_3$ and thus $a_3 \mathcal{R}_3 b_3$. Since $\langle b_1, b_3 \rangle \in \min J_{1,3}$ it follows that $\langle a_1, a_3 \rangle \in \min J_{1,3}$.

Corollary 17 Proposition 18 Suppose that $\langle a_1, a_2 \rangle \in \min J_{1,2}$, $\langle a_2, a_3 \rangle \in \min J_{2,3}$, not $a_1 \mathcal{R}_1 \bot$ and not $\mathcal{R}_2 a_2$. Then $a_2 \mathcal{R}_2 \langle a_1, a_3 \rangle$.

Proof. WG (a1, a2, B1) and SC (a3, a2, B3) and $\langle a_1, a_3 \rangle \in J_{1,3}$, hence $a_2 \mathcal{R}_2 \langle a_1, a_3 \rangle$. Note that $\langle a_1, a_3 \rangle \in \min J_{1,3}$, but this is not used in the proposition.
Theorem 19 Suppose that \( \langle a_1, a_3 \rangle \in \min J_{1,3} \) then there is \( a_{2,1}, a_{2,2} \in B_2 \) such that \( \langle a_{1, a_2} \rangle \in \min J_{1,2} \) and \( \langle a_{2,2}, a_3 \rangle \in \min J_{2,3} \) and \( a_{2,1} R_{2} a_{2,2} \).

Proof. Since \( \langle a_1, a_3 \rangle \in J_{1,3} \) then there is \( b_3 \in B_2 \) such that \( \langle a_1, b_3 \rangle \in J_{1,2} \) and \( \langle b_2, a_3 \rangle \in J_{3,2} \). Since \( J_{1,2} \) and \( J_{3,2} \) satisfy connectivity it holds that \( \langle c_1, c_2 \rangle \in \min J_{1,2} \) and \( \langle c_1, c_2 \rangle \leq \langle a_1, b_3 \rangle \) and that \( \langle d_2, d_3 \rangle \in \min J_{2,3} \) and \( \langle d_2, d_3 \rangle \leq \langle b_2, a_3 \rangle \). Hence, \( a_1 R_{1} c_1 \) and \( c_2 R_{2} d_2 \) and \( c_2 R_{2} d_2 \) and \( d_2 R_{3} a_3 \). Since \( \langle c_1, c_2 \rangle \in J_{1,2} \) and \( c_2 R_{2} b_2 \) and \( b_2 R_{3} a_3 \) then \( \langle c_1, a_3 \rangle \in J_{1,3} \). From this and \( \langle a_1, a_3 \rangle \in J_{1,3} \) follows according to proposition 12 that \( c_1 R_{1} a_1 \). Since \( a_1 R_{1} c_1 \) it follows that \( a_1 Q_{1} c_1 \). Since \( \langle c_1, c_2 \rangle \in \min J_{1,2} \) it follows that \( \langle a_1, c_2 \rangle \in \min J_{1,2} \). Choose \( a_{2,1} \) as \( c_2 \).

Since \( \langle a_1, b_3 \rangle \in J_{1,2} \) and \( b_2 R_{2} d_2 \) and \( d_2, d_3 \rangle \in J_{2,3} \) then \( \langle a_1, d_3 \rangle \in J_{1,3} \). From this and \( \langle a_1, d_3 \rangle \in \min J_{1,3} \) follows that \( a_3 R_{3} d_3 \). Since \( a_3 R_{3} d_3 \) it follows that \( a_3 Q_{3} d_3 \). Since \( \langle d_2, d_3 \rangle \in \min J_{2,3} \) it follows that \( \langle d_2, a_3 \rangle \in \min J_{2,3} \). Choose \( a_{2,2} \) as \( d_3 \). Since \( c_2 R_{2} b_2 \) and \( b_2 R_{3} d_2 \) it follows that \( c_2 R_{2} d_2 \), thus \( a_{2,1} R_{2} a_{2,2} \). ■

Corollary 20 Suppose that \( \langle a_1, a_3 \rangle \in \min J_{1,3} \), not \( a_1 R_{1} a_1 \) and not \( T R_{3} a_3 \). Then there is \( a_2 \in B_2 \) such that \( a_2 \春秋 (a_1, a_3) \).

Proof. According to the theorem there is \( a_{2,1}, a_{2,2} \) such that \( \langle a_1, a_2, b \rangle \in \min J_{1,2} \) and \( \langle a_{2,2}, a_3 \rangle \in \min J_{2,3} \) and \( a_{2,1} R_{2} a_{2,2} \). Hence, \( W G_{5} (a_1, a_2, B_1) \) and \( S C_{5} (a_3, a_{2,2}, B_3) \). Since \( a_{2,1} R_{2} a_{2,2} \) it follows that \( \langle a_{2,1}, a_3 \rangle \in J_{2,3} \). Suppose that \( \langle a_{2,1}, b_3 \rangle \in J_{2,3} \). Since \( \langle a_{2,1}, b_3 \rangle \in J_{1,2} \) and \( J_{1,3} = J_{1,2} \cup J_{2,3} \) it follows that \( \langle a_1, b_3 \rangle \in J_{1,3} \). Since \( \langle a_1, a_3 \rangle \in \min J_{1,3} \) it follows that \( a_3 R_{3} b_3 \). This shows that \( S C_{5} (a_3, a_{2,1}, B_3) \). Hence, \( a_{2,1} \春秋 (a_1, a_3) \).

To prove that \( a_{2,2} \春秋 (a_1, a_3) \) we must prove that \( W G_{5} (a_1, a_{2,2}, B_1) \). Since \( a_{2,1} R_{2} a_{2,2} \) it follows that \( \langle a_1, a_{2,2} \rangle \in J_{1,3} \). Suppose that \( \langle b_1, a_{2,2} \rangle \in J_{1,2} \). Then \( \langle b_1, a_3 \rangle \in J_{1,3} \) and since \( \langle a_1, a_3 \rangle \in \min J_{1,3} \) it follows according to proposition 12 that \( b_1 R_{1} a_1 \). This shows that \( W G_{5} (a_1, a_{2,2}, B_1) \). Hence, \( a_{2,2} \春秋 (a_1, a_3) \). ■

The proof of the corollary shows that there can be more than one intervenient corresponding to the same minimal joining from \( B_1 \) to \( B_3 \).

Proposition 21 Suppose that \( a_2 \春秋 (a_1, a_3) \in \min J_{1,3} \) and \( b_2 \春秋 (b_1, b_3) \in \min J_{1,3} \) and not \( a_1 \land b_1 R_{1} a_1 \) and not \( T R_{3} a_3 \lor b_3 \). Then the following holds:

1. If \( \langle a_1 \land b_1, a_3 \land b_3 \rangle \in \min J_{1,3} \) then \( a_2 \land b_2 \春秋 \langle a_1 \land b_1, a_3 \land b_3 \rangle \).

2. If \( \langle a_1 \lor b_1, a_3 \lor b_3 \rangle \in \min J_{1,3} \) then \( a_2 \lor b_2 \春秋 \langle a_1 \lor b_1, a_3 \lor b_3 \rangle \).

Proof. We first prove (1). Note that \( \langle a_1 \land b_1, a_2 \land b_2 \rangle \in J_{1,2} \). Note further that \( W G_{5} (a_1, a_2, B_1) \) and \( W G_{5} (b_1, b_2, B_1) \), which implies that \( W G_{5} (a_1 \land b_1, a_2 \land b_2, B_1) \) according to proposition 5 (vi) and \( a_3 \land b_1 \) is not degenerated. From \( \langle a_2, a_3 \rangle \), \( \langle b_2, b_3 \rangle \in J_{2,3} \) it follows that \( \langle a_2 \land b_2, a_3 \land b_3 \rangle \in J_{2,3} \). Suppose that \( \langle a_2 \land b_2, c_3 \rangle \in J_{2,3} \). Since \( J_{1,3} = J_{1,2} \cup J_{2,3} \) it follows that \( \langle a_1 \land b_1, c_3 \rangle \in J_{1,3} \). Then from \( \langle a_1 \land b_1, a_3 \land b_1 \rangle \in \min J_{1,3} \) follows according to proposition 12 that \( a_3 \land b_3 R_{3} c_3 \), and hence, \( S C_{5} (a_3 \land b_3, a_2 \land b_2, B_3) \). Thus \( a_2 \land b_2 \春秋 \langle a_1 \land b_1, a_3 \land b_3 \rangle \).
We now prove (2). Note that \( \langle a_2 \lor b_2, a_3 \lor b_3 \rangle \in J_{2,3} \), and further, SC \((a_1, a_2, B_1)\) and SC \((b_3, b_2, B_2)\), which implies that SC \((a_3 \lor b_3, a_2 \lor b_2, B_3)\) according to proposition 5 (vii) and \( a_3 \lor b_3 \) is non-degenerated. From \( \langle a_2, a_3 \rangle, \langle b_2, b_3 \rangle \in J_{2,3} \) it follows that \( \langle a_2 \lor b_2, a_3 \lor b_3 \rangle \in J_{2,3} \). Suppose that \( \langle c_1, a_2 \lor b_2 \rangle \in J_{1,2} \). Since \( J_{1,3} = J_{1,2} | J_{2,3} \) it follows that \( \langle c_1, a_3 \lor b_3 \rangle \in J_{1,3} \). Then from \( \langle a_1 \lor b_1, a_3 \lor b_3 \rangle \in \min J_{1,3} \) follows \( c_1 R_1 a_1 \lor b_1 \) according to proposition 12, and hence, WG \((a_1 \lor b_1, a_2 \lor b_2, B_1)\).

Thus \( a_2 \lor b_2 \sim (a_1 \lor b_1, a_3 \lor b_3) \).

**Proposition 22** Suppose \( a_2 \sim (a_1, a_3) \in \min J_{1,3} \) and \( b_2 \sim (b_1, b_3) \in \min J_{1,3} \) and, furthermore, not \( a_1 \land b_1 R_1 \perp \) and not \( \top R_3 a_3 \lor b_3 \). Then there are \( c_2, d_2 \in B_2, c_3 \in B_3 \) and \( d_1 \in B_1 \) such that \( c_2 \sim (a_1 \land b_1, c_3) \in \min J_{1,3} \) and \( d_2 \sim (d_1, a_3 \lor b_3) \in \min J_{1,3} \).

**Proof.** (1) From \( \langle a_1, a_3 \rangle, \langle b_1, b_3 \rangle \in \min J_{1,3} \) it follows according to theorem 14 that there is \( c_3 \in B_3 \) such that \( \langle a_1 \land b_1, c_3 \rangle \in \min J_{1,3} \). From \( \langle a_2, a_3 \rangle \in J_{2,3} \) and \( \langle b_2, b_3 \rangle \in J_{2,3} \) it follows that \( \langle a_2 \land b_2, a_3 \land b_3 \rangle \in J_{2,3} \). Since \( \langle a_1 \land b_1, c_3 \rangle \in \min J_{1,3} \) and \( J_{1,3} = J_{1,2} | J_{2,3} \) it follows that there is \( c_2 \in B_2 \) such that \( \langle a_1 \land b_1, c_2 \rangle \in J_{1,2} \) and \( \langle c_2, c_3 \rangle \in J_{2,3} \). Suppose that there is \( x_1 \in B_1 \) such that \( \langle x_1, c_2 \rangle \in J_{1,2} \). Then \( \langle x_1, c_3 \rangle \in J_{1,3} \) and since \( \langle a_1 \land b_1, c_3 \rangle \in \min J_{1,3} \) it follows that \( x_1 R_1 a_1 \land b_1 \). This shows that WG \((a_1 \land b_1, c_2, B_1)\).

Suppose that \( \langle c_2, x_3 \rangle \in J_{2,3} \). Since \( \langle a_1 \land b_1, c_2 \rangle \in J_{1,2} \) it follows that \( \langle a_1 \land b_1, x_3 \rangle \in J_{1,3} \) and this implies, since \( \langle a_1 \land b_1, c_3 \rangle \in \min J_{1,3} \), that \( c_3 R_3 x_3 \). This shows, together with \( \langle c_2, c_3 \rangle \in J_{2,3} \), that SC \((c_3, c_2, B_3)\), and since not \( a_1 \land b_1 R_1 \perp \) it follows that \( c_2 \sim (a_1 \land b_1, c_3) \in \min J_{1,3} \).

(2) is proved analogously. From \( \langle a_1, a_3 \rangle, \langle b_1, b_3 \rangle \in \min J_{1,3} \) it follows according to theorem 14 that there is \( d_3 \in B_3 \) such that \( \langle a_1, a_3 \lor b_3 \rangle \in \min J_{1,3} \). From \( \langle a_1, a_2 \rangle \in J_{1,2} \) and \( \langle b_1, b_2 \rangle \in J_{1,2} \) it follows that \( \langle a_1 \lor b_1, a_2 \lor b_2 \rangle \in J_{1,2} \). Since \( \langle d_1, a_3 \lor b_3 \rangle \in \min J_{1,3} \) and \( J_{1,3} = J_{1,2} | J_{2,3} \) it follows that there is \( d_2 \in B_2 \) such that \( \langle d_1, d_2 \rangle \in J_{1,2} \) and \( \langle d_2, a_3 \lor b_3 \rangle \in J_{2,3} \). Suppose that there is \( y_3 \in B_3 \) such that \( \langle d_2, y_3 \rangle \in J_{2,3} \). Then \( \langle d_1, y_3 \rangle \in J_{1,3} \) and since \( \langle d_1, a_3 \lor b_3 \rangle \in \min J_{1,3} \) it follows that \( a_3 \lor b_3 R_3 y_3 \). This shows that SC \((a_3 \lor b_3, d_2, B_2)\).

Suppose that \( \langle y_1, d_2 \rangle \in J_{1,2} \). Since \( \langle d_2, a_3 \lor b_3 \rangle \in J_{2,3} \) it follows that \( \langle y_1, a_3 \lor b_3 \rangle \in J_{1,3} \) and this implies, since \( \langle d_1, a_3 \lor b_3 \rangle \in \min J_{1,3} \), that \( y_1 R_1 d_1 \). This shows, together with \( \langle d_1, d_2 \rangle \in J_{1,2} \), that WG \((d_1, d_2, B_1)\), and since not \( \top R_3 a_3 \lor b_3 \) it follows that \( d_2 \sim (d_1, a_3 \lor b_3) \in \min J_{1,3} \).

Due to the importance of minimality emphasized above, note that the following holds given that \( J_{1,3} = J_{1,2} | J_{2,3} \). Suppose \( C_2 \) is a subset of \( B_2 \) consisting of correspondence-minimal interventions from \( B_1 \) to \( B_3 \) and it holds that if \( \langle a_1, a_3 \rangle \in \min J_{1,3} \) then there is \( c_2 \in C_2 \) such that \( c_2 \sim (a_1, a_3) \). Then

\[
J_{1,3} = \{ \langle a_1, a_3 \rangle \in B_1 \times B_3 \mid \exists c_2 \in C_2 : \langle a_1, c_2 \rangle \in J_{1,2} \text{ and } \langle c_2, a_3 \rangle \in J_{2,3} \}.
\]

---

1. Note that \( \langle a_1 \land b_1, a_2 \land b_2 \rangle \in J_{1,2} \) and WG \((a_1, a_2, B_1)\) and WG \((b_1, b_2, B_1)\), which implies that WG \((a_1 \land b_1, a_2 \land b_2, B_1)\) according to proposition 5 (vi).
Hence, a set of correspondence-minimal interventional can be a convenient way for characterizing a set of joinings.

Note that if \((a_1, a_3) \in \min J_{1,3}\) there can be several interventional that correspond to \((a_1, a_3)\).

**Proposition 23** If and only if \((a_1, a_3) \in \min J_{1,3}\) and \(c_2\) is the only element up to \(Q_2\)-similarity in \(1v_2(B_2, B_1, B_3)\) such that \(c_2 \cap (a_1, a_3)\), then \(c_2\) is ground-minimal and consequence minimal.

**Proof.** (1) Suppose that \((a_1, a_3) \in J_{1,3}\). Then there is \((b_1, b_3) \in \min J_{1,3}\) such that \((b_1, b_3) \leq (a_1, a_3)\). Suppose not \(b_1 \not\perp\) and not \(\top R_2 b_2\). According to 20 there is \(c_2 \in B_2\) such that \(c_2 \cap (b_1, b_3)\). Hence, \((b_1, c_2) \in J_{1,2}\) and \((c_2, b_3) \in J_{2,3}\), which together with \(a_1 R_1 b_1\) and \(b_3 R_3 a_3\) implies \((a_1, c_2) \in J_{1,2}\) and \((c_2, a_3) \in J_{2,3}\). Since \(J_{1,3} = J_{1,2} \cup J_{2,3}\) it follows that \((a_1, a_3) \in J_{1,3}\).

(2) Suppose \((a_1, a_3) \in B_1 \times B_3\) such that there is \(c_2 \in B_2\) such that \((a_1, c_2) \in J_{1,2}\) and \((c_2, a_3) \in J_{2,3}\). Since \(J_{1,3} = J_{1,2} \cup J_{2,3}\) it follows that \((a_1, a_3) \in J_{1,3}\). ■
References


REFERENCES


Odelstad, J. (1989) Om den metodologiska subjektivismen. Lecture delivered in the higher seminar of theoretical philosophy, Uppsala University, April 1989. (Unpublished manuscript in Swedish.)


