



Negotiating different disciplinary discourses: biology students' ritualized and exploratory participation in mathematical modeling activities

Olov Viirman^{1,2}  • Elena Nardi³

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Abstract

Non-mathematics specialists' competence and confidence in mathematics in their disciplines have been highlighted as in need of improvement. We report from a collaborative, developmental research project which explores the conjecture that greater integration of mathematics and biology in biology study programs, for example through engaging students with Mathematical Modeling (MM) activities, is one way to achieve this improvement. We examine the evolution of 12 first-semester biology students' mathematical discourse as they engage with such activities in four sessions which ran concurrently with their mandatory mathematics course and were taught by a mathematician with extensive experience with MM. The sessions involved brief introductions to different aspects of MM, followed by small-group work on tasks set in biological contexts. Our analyses use the *theory of commognition* to investigate the tensions between *ritualized* and *exploratory participation* in the students' MM activity. We focus particularly on a quintessential *routine* in MM, *assumption building*: we trace attempts which start from ritualized engagement in the shape of "guesswork" and evolve into more productively exploratory formulations. We also identify signs of persistent *commognitive conflict* in the students' activity, both *intra-mathematical* (concerning what is meant by a "math task") and *extra-mathematical* (concerning what constitutes a plausible solution to the tasks in a biological sense). Our analyses show evidence of the fluid interplay between ritualized and exploratory engagement in the students' discursive activity and contribute towards what we see as a much needed distancing from operationalization of the commognitive constructs of ritual and exploration as an unhelpfully dichotomous binary.

Keywords University mathematics education · Discourse · Mathematics in biology · Commognition · Rituals and explorations · Assumption building

✉ Olov Viirman
olov.viirman@hig.se

1 Teaching mathematics to biology students through mathematical modeling

Research into the mathematical needs of non-mathematics specialists is by no means new (e.g., Kent & Noss, 2003). Participants in university-level studies are often non-mathematics specialists (e.g., engineers or pre-service teachers), but their specialism often remains a mere part of the study's backdrop (Biza, Giraldo, Hochmuth, Khakbaz, & Rasmussen, 2016). The small but growing number of studies in this area have touched on issues such as the double discontinuity (Klein, 1908/1931) between school, university, and workplace mathematics; the challenges of teaching mathematical modeling at school and university levels; issues of confidence in, and appreciation for, mathematics; and embeddedness of mathematics into other disciplines (Biza et al., 2016; Gould, Murray, & Sanfratello, 2012).

Within biology, the increasing importance of mathematics is placing new demands on the education of future biologists. In the USA, for example, two national projects have focused on developing undergraduate biology education, largely through a greater integration of mathematics and biology in the curriculum (Brewer & Smith, 2011; Steen, 2005). However, a potential problem with placing greater emphasis on mathematics in biology education is that "biology education is burdened by habits from a past where biology was seen as a safe harbour for math-averse science students" (Steen, 2005, p. 14). For instance, in Norway, where the research we report in this paper is carried out, biology departments are struggling with attracting students to mathematics courses beyond those mandatory to the biology programs. The project that we report from this paper grew out of this struggle and explores the conjecture that mathematical modeling (MM) can be a vehicle for the integration of mathematics and biology (Brewer & Smith, 2011) and that engagement with MM activities can contribute to more positive attitudes towards, and competence in, both biology and mathematics (Chiel, McManus, & Shaw, 2010).

In the wake of, for instance, the *Vision and Change* program (Brewer & Smith, 2011) the use of MM in university biology education has received some attention in the literature, but mostly in the form of descriptive reports of modules or activities involving biology-related MM (e.g., Ginovart, 2014; Powell, Kohler, Haefner, & Bodily, 2012; Volpert, 2011). On the other hand, in the context of an advanced graduate-level biological modeling course aimed at both applied mathematics and biology students, Smith, Haarer, and Confrey (1997) present an analysis of how applied mathematicians and biologists understand dynamic population models from their respective professional perspectives, and how these models come to serve as bridges between these perspectives. For the mathematics students, "an important aspect of constructing an algebraic model was that it be *mathematically analyzable*" (Smith et al., 1997, p. 451; emphasis in the original), whereas the biology students were concerned with the model incorporating as much of the complexity of the biological setting as possible. This differentiation reflects the distinctions Smith et al. (1997) make: first, between "theoretical" and "descriptive models" (p. 461); second, between "ad hoc" and "first principles models" (p. 464). Theoretical models build on the biological understanding of the underlying processes, whereas descriptive models aim to describe observed phenomena as accurately as possible. The second distinction concerns how the mathematics is applied: an ad hoc model is built so that it fits the data, for instance through curve fitting, whereas a first principles model is built so that it fits known mathematical relationships (for instance exponential growth) to a biological setting.

Smith et al. also present a classification of five different epistemological roles mathematical models can play in biology: a "caricature" of a biological system can be used to explore

dynamics of interaction; modeling as “experiment” uses a variety of models to accumulate information; a “first principles” model uses the relationship between the biological system and the mathematical structure; an “empiricist” model suggests what data needs to be collected; and a “dialectic” model uses the relationship between results from the model and former biological knowledge to form the basis for developing new biological knowledge (Smith et al., 1997, p. 467). In the same vein, Svoboda and Passmore (2013) argue that much educational research on MM has failed to take into account the different epistemic aims of MM and argue that, beyond explanation and prediction, biological models can also serve the aims of understanding and exploring complex phenomena, as well as developing biological concepts. Simple, “false” (Smith et al., 1997, p. 124) models can help biologists structure their understanding of complex biological phenomena and identify what it is that they do not yet know. Moreover, components added to the model for mathematical reasons can often be interpreted in biological terms, and thus contribute to biological concept development.

In this paper, we report from a project in which we develop biology-related MM activities and introduce them to year 1 biology students with the awareness of two potential caveats: that this engagement, often requiring the synthesis and application of mathematical knowledge gained elsewhere, might require a certain level of academic experience (Edelstein-Keshet, 2005); and, that an integrative approach to mathematics and biology education might have also adverse effects, such as breadth at the expense of depth, or mathematics anxiety problems (Madlung, Bremer, Himelblau, & Tullis, 2011). We examine the evolution of the participating biology students’ engagement with mathematics as they grapple with MM activities designed to integrate elements of mathematics and biology. In particular, we focus on a quintessential aspect of engaging with MM, the processes of constructing the assumptions which underpin the model (Maaß, 2006), hitherto labeled *assumption building*.

The process of MM is often described with reference to the Modeling Cycle (e.g., Blum, Galbraith, Henn, & Niss, 2007). In its simplest form, this cycle involves taking a problem from an extra-mathematical domain (for instance, biology), translating it into mathematical terms, solving it through mathematical means, and then translating the solution back into the terms of the extra-mathematical domain. In this process, assumption building plays a central role. Indeed, it is highlighted as one of the necessary MM competencies (Maaß, 2006) even though it is yet to become the focus of extensive research, particularly at university level. However, some work has been done emphasizing the complexities of assumption building and problematizing the role often assigned to it. For instance, Galbraith and Stillman (2001) alert us to the perils of “oversimplify[ing] the role of assumptions by consigning them to some visible phase of a heuristic process such as model formulation” (p. 301). They distinguish between three types of assumptions, based on the role they play in the modeling process: assumptions associated with model formation, assumptions associated with mathematical processing, and assumptions associated with strategic choices in the solution process (Galbraith & Stillman, 2001, p. 305). Furthermore, they also discuss the role of assumptions involved in students’ sense making concerning the tasks offered to them. Djepaxhija, Vos, and Fuglestad (2015) denote these as “assumptions about task expectations” (p. 852), emerging from the learning environment. They observe that such assumptions are epistemologically different from those listed by Galbraith and Stillman and typically have their basis in “a school culture with unwritten rules” (Djepaxhija et al., 2015, p. 852). We take the observations on the importance and complexity of assumption building as cues in our analyses, which aim to add further nuanced insight into this important form of reasoning—mathematical and other—that students draw on as they engage with MM activities.

We note that numerous studies have investigated students' emerging reasoning as they engage with, for example, problem-solving activity and that this reasoning is often described using binaries such as *instrumental* and *relational understanding* (Skemp, 1976) or *creative* and *imitative reasoning* (Lithner, 2008). In our analyses, however, we take an alternative view: we study the students' experience through what they say and do as they engage with MM activities; we study, in other words, the evolution of their discourse. To do so, we study patterns in their activity, known in the language of the discursive perspective we espouse in our analysis—the theory of commognition (Nardi, Ryve, Stadler, & Viirman, 2014; Sfard, 2008, p. 183–5)—as *routines* of a *ritualized* or *exploratory* kind. The episodes we present here therefore illustrate various forms of, and tensions between, *ritualized* and *exploratory participation* in the students' MM activity.

In what follows, we introduce those components of the commognitive perspective pertinent to the data analysis we present in this paper and formulate the research questions that the paper investigates in the light of these components. We then introduce the study's aims, participants, and methods of data collection and analysis and give an outline of the four MM sessions that are at the heart of the data accounts and analysis that follow. We conclude with a brief discussion of substantive implications (what do these analyses tell us about these students' evolving experience of mathematics in the context of biology?), theoretical implications (what do these analyses indicate about the distinction, interplay, and tensions between ritualized and exploratory participation in discursive activity?), and practical implications (what do these analyses tell us about the potency of MM-focused pedagogy in biology courses?) of our analysis.

2 Commognitive conflict and ritualized/exploratory participation in MM activities

According to the commognitive perspective, 'it is by reproducing familiar communicational moves in appropriate new situations that we become skillful discursants and develop a sense of meaningfulness of our actions' (Sfard, 2008, p. 195). If mathematical learning is initiation into the discursance of mathematics, then it generally involves substantial discursive shifts for learners—and the teaching of mathematics involves facilitating such shifts. Communication through written or spoken language and manipulation of physical objects and artifacts are the main means to the discursive ends of teaching and learning. Consequently, a discourse is distinguished by a community's *word use*, *visual mediators*, *endorsed narratives*, and *routines* (Sfard, 2008, p. 133–135). Specifically, a *routine* is a set of meta-rules that describe a repeated discursive action. Sfard defines three types of mathematical routines: *explorations*, *deeds*, and *rituals*. *Explorations* are *routines* "whose performance counts as completed when an endorable narrative is produced or substantiated" (p. 224), for example defining, proving, or equation solving, whereas *deeds* are *routines* that involve practical action, resulting in change in objects, either primary or discursive (p. 236), such as calculating or measuring. Additionally, some routines "begin their life as neither *deeds* nor *explorations* but as *rituals*, that is, as sequences of discursive actions whose primary goal [...] is neither the production of an endorsed narrative nor a change in objects, but creating and sustaining a bond with other people" (p. 241). For Sfard, *rituals* are a "natural, mostly inevitable, stage in routine development" (p. 245). Another important aspect of *routine* development is *commognitive conflict*, which occurs when discursant(s) act according to diverging or even conflicting discursive

rules. This is often a necessary feature of discursive change, and hence of learning (p. 256), which takes place at *object-level* (involving cumulative extension of endorsed narratives about already constructed mathematical objects) and at *meta-level* (involving changes in meta-discursive rules).

Of particular relevance to the analyses we present in this paper is what Heyd-Metzuyanin, Tabach, and Nachlieli (2016) note as a hitherto relatively under-researched aspect of the commognitive framework, namely the characterization of the relationship between *rituals* and *explorations*. As Lavie, Steiner, and Sfard (2018) note, also in this issue, rituals may morph into explorations as a learner's performance shifts from being process-oriented to becoming outcome-oriented. De-ritualization may involve manifestations of flexibility and broadened applicability—or, in Lavie et al.'s terms, *vertical* (a new step in a procedure building on outcomes of previous steps) or *horizontal* (a procedure being conducted in a number of alternative ways which generate the same output) *bonding*.

Our analyses aim to contribute to the study of what may facilitate said de-ritualization. We see our analysis as affording opportunities for investigating mathematical reasoning—often discussed in terms of binaries such as the aforementioned distinction between *creative* and *imitative reasoning* (Lithner, 2008)—from what we see as a more encompassing, fluid discursive perspective. We also aim to contribute to a stronger operationalization of the commognitive constructs of *ritual* and *exploration*. Our analyses reinforce what we see as much needed distancing from their operationalization as yet another binary. Particularly, we aim to do so in a cross-disciplinary context, with the challenges of navigating across different disciplinary discourses that this context entails.

In the light of above embedding of our analysis in the commognitive perspective, the research questions that this paper explores are as follows: What evidence of ritualized and exploratory engagement with assumption building can be traced in biology students' engagement with MM activities? What evidence of commognitive conflict can be traced in this engagement, especially in the students' efforts to navigate across the discourses of mathematics and biology?

Before presenting our data analysis, we introduce the study's context, aims, methods, and participants and outline the MM activities that were the backbone of our data collection.

3 Engaging year 1 biology students with MM activities

3.1 Context, participants, and data collection

The project comprises cycles of developmental activity (planning, implementation, reflection, feedback) which are theoretically informed, contribute to the emergence of theory, and take place in a partnership between teachers (in this case, a university mathematician) and mathematics education researchers. The teaching took place at a large, research-intensive Norwegian university where biology students take a compulsory mathematics course in the first semester of their university studies, designed not specifically for the biology undergraduate program but for students from about 20 different natural science programs. There is little collaboration between the mathematics and biology departments, and few opportunities for focusing on issues specific to biology in the mathematics course.

This paper concerns the first cycle of the project, consisting of one 3-h pilot session with 10 volunteering students in the spring, and a main sequence of 3-h sessions in the following

autumn, where we met a different group of 12 volunteering students (out of a cohort of approximately 100) on four occasions. All students on the Biology program have taken mathematics up to a level in upper secondary school known in the Norwegian system as R1 or S2,¹ and the 12 volunteers reflect the demographic makeup of the total cohort concerning gender and educational background. To the best of our knowledge, the students had no prior experience of mathematical modeling and their prior experience of biology, beyond the elementary introduction that constitutes part of the Norwegian curriculum, was merely what they had so far covered in the introductory first-semester biology course running in parallel with the mathematics course. The research team comprised three mathematics education researchers and one research mathematician with extensive experience of MM. None had extensive prior experience with teaching MM but we note that the mathematician who led the sessions has substantial experience in teaching applied mathematics.

In this paper, we draw on data from the autumn sequence of four sessions, which took place concurrently with the students' mandatory first-semester mathematics course. The aim of these sessions was not primarily teaching the students new mathematics, or even principles of MM, but rather letting them experience how mathematics and particularly MM can be relevant for addressing biological problems, and whetting their appetites for studying more mathematics. All sessions consisted of brief lectures introducing various aspects of MM, followed by group work on MM tasks set in a biological context. There were three groups labeled A, B, and C, with students in the groups labeled as A1, A2, etc. The teaching was conducted in English, but most student group work and student contributions to group discussions were in Norwegian. Before describing the process of data reduction and analysis, we present an overview of the four sessions.

3.2 Overview of the four sessions

The first session began with an introduction to MM and to the modeling cycle (Blum et al., 2007). The lecturer emphasized the role of *assumption building*, namely making simplifying or clarifying assumptions as a part of the translation of a problem in biology into the mathematical domain: "assumptions are vital to forming a sensible model". He highlighted three functions that assumptions serve: to *simplify* the model, to *define relationships between variables* when precise relationships are not known, and to *determine the values of parameters* when exact values are not known. After a brief warm-up problem, most of the session was spent on the *Roadkill Rabbits* problem, a problem which can be seen as providing an opportunity to experience broadly some of the epistemological roles of MM in biology described by Smith et al. (1997), namely the "experiment" and the "caricature":

"Driving across Nevada, you count 97 dead but still easily recognizable jackrabbits on a 200-km stretch of Highway 50. Along the same stretch of highway, 28 vehicles passed you going the opposite way. What is the approximate density of the rabbit population to which the killed ones belonged?"

We note that our aim was not so much for the students to produce a fully functioning model as to engage them in assumption building routines, while requiring few computational routines

¹ R1 is the first course of two for the natural science program, whereas S2 is the second course of two for the social sciences program. Content includes elementary differential calculus, but not, for instance, integrals or differential equations.

beyond basic arithmetic, and to see whether they were able to use the given information, as well as everyday experience, towards productive engagement with the problem. It involves all three aforementioned functions of assumptions. Simplifying assumptions include, for instance, assuming constant population density and traffic intensity. Undetermined values of parameters include, for instance, the speed of the cars and the rabbits. Parts of the students' work on this problem are discussed in Sect. 4.1. See also the lecturer's preferred solution in the [Appendix](#).

The bulk of the second session concerned using MM to model change. The lecturer introduced a problem concerning *Yeast Growth* in a Petri dish which, contrary to the very open *Roadkill Rabbits* problem, was broken into sub-problems that the students worked on for 10–15 min each, with whole-class summaries in between.

The third session began with a further problem on the modeling of change, this time concerned with modeling the decay in the body of *Digoxin*, a drug used to treat heart disease. Detailed descriptions of the *Yeast Growth* and *Digoxin* problems, as well as an analysis of the students' work on these problems are in (Viirman & Nardi, 2017; Viirman & Nardi, 2018). After short lectures on non-linear models and modeling using geometric similarity, the *Terror Bird* problem was introduced. In this problem, the students were asked to estimate the weight of an extinct species of bird, based on measurements of fossilized femur bones and present-day data on the relation between femur circumference and body weight in various bird species.

Finally, the bulk of the fourth and final session was devoted to the *Rabbits and Foxes* problem, concerning the dynamics of the interaction between two populations, foxes and rabbits. We note that this problem can be seen as providing an opportunity to experience broadly what Smith et al. (1997) describe as the “first principles” and the “caricature” epistemological roles of MM in biology:

“Let R_n denote the population of the rabbits at time period n (in months) and F_n denote the population of predator foxes at time period n . Suggest a reasonably simple mathematical model based on one or several difference equations, which would describe the competition between the two species. Suggest a strategy (or several) which can be used to analyze efficiently the long-term behavior of the two species and to learn about the outcome of the competition.”

Here, the notion of assumption building was again central. The students were initially provided with several simplifying assumptions:

1. There is enough food for rabbits and the population of rabbits increases by a constant rate. That is, the rabbit population increases exponentially.
2. The population of rabbits decreases as a result of the interactions between rabbits and foxes.
3. The rabbits are the only source of food for foxes. Therefore, in the absence of rabbits, the population of foxes decreases by a constant rate and dies out. That is, the fox population decreases exponentially.
4. The population of foxes increases as a result of interactions between rabbits and foxes.
5. The rabbits and foxes live in a closed environment. This means that there is no interaction between these two species and other species, there is no emigration/immigration from/to the forest, and there is no harvest or hunting.

We point out that, based on observations from the pilot session, the decision was made that researchers would act as minimally engaging in the students' work on the tasks. They were,

however, available for answering student questions at all times during the sessions. Still, the students asked very few questions, somewhat surprisingly, given the difficulties they faced with many of the tasks. As the sessions progressed, the students' need of support became increasingly evident. Hence, for the fourth session, contrary to the first three sessions, a member of the research team joined each of the groups to provide support if needed. This support consisted both in answering any questions the students might have concerning the interpretation of the task, but also in providing hints when they seemed to hit a stumbling block. Parts of the students' work on the *Rabbits and Foxes* problem are discussed in Sect. 4.2. See also the lecturer's preferred solution in the [Appendix](#).

Before zooming in on a selection of episodes from the students' work on the *Roadkill Rabbits* (4.1) and *Rabbits and Foxes* (4.2) problems, we outline the analytical process that led to the selection of these episodes.

3.3 From raw data to a selection of 'Roadkill Rabbits' and 'Rabbits and Foxes' episodes

All whole-group and small-group activity during the sessions was video and audio recorded, and then transcribed. In addition, much of the written material produced by the students was collected. As working with condensed accounts makes potential patterns in the discursive activity more easily discernible, the first author, who had been present at all four sessions, first produced descriptive accounts of these sessions. Both authors then scrutinized these accounts for evidence of students' routine (ritualized and exploratory) engagement with the problems, cataloging episodes where one or more students focused on one particular routine. These included routines actually engaged in, such as graph construction, as well as routines talked about but not performed, such as data collection through observation or sampling. We looked at to what extent the discourse was "objectified and alienated" (Sfard, 2008, p.42–44): for instance, whether the students talked about mathematics as performing operations on symbols, or in terms of properties of mathematical objects. We also explored what aims the routine seemed to serve: could we, for example, see signs in the students' discourse of engaging with the routine out of a sense of its relevance for solving the problem, or was it performed mainly because the students were expected to do so (out of obligation to the research team)? Further signs of ritualized routine use could be, for instance, a strong reliance on external sources for substantiation, rigid rule following and mimicking previously encountered routines without regard for relevance to the problem at hand. To clarify how we conducted the cataloging of episodes, we present examples from the data of ritualized and exploratory routine use.

Examples of mathematical routine use that we have classified as ritualized include, for instance, the algebraic construction routine engaged in by group B in session 4 (see Sect. 4.2), which was solely concerned with how to manipulate the symbols, and the graph construction of group B in session 2 (Viirman & Nardi, 2017), where a previously established routine was employed despite the students realizing that it did not help them in solving the task at hand. On the other hand, student C2, working on the *Digoxin* problem in session 3 (Viirman & Nardi, 2018), used the graph he had plotted to argue that the proportionality constant could not be positive (as it was mistakenly said in the problem formulation). We classified this as exploratory routine use, since he used his own work for substantiation purposes rather than relying on external authority.

Concerning ritualized and exploratory biological routine use, we stress that we mean biological discourse in a broad sense. Since the participants in the study were first-semester

students, they had not actually studied much biology yet. Still, they often used arguments from biology in their reasoning about the problems, but this biological reasoning relied as much on “colloquial” (Sfard, 2008, p.132) insights into (for example) the behavior of certain species as on more “literate” (p.132) elements of scientific biological discourse. We give concrete examples in 4.1 and 4.2. We note, however, that we saw less of ritualized biological routine use. This is not surprising, given that the students were at the early stages of their biological studies and that, as ritual participation is largely about social acceptance in the discourse community, the setting of this study was a course in mathematics (that students of biology had to take) rather than a course in biology.

While working with the descriptive accounts, focusing on routine use, we noticed how the students’ engagement with *assumption building* routines required them to *navigate across the discourses of biology and mathematics* as well as colloquial discourse. Subsequently, the first author returned to the raw data and produced preliminary analytical accounts of the sessions with a focus on the assumption-building theme. These accounts included data excerpts of all episodes (25) relevant to the theme, together with a preliminary analysis, including, for instance, the previously produced classification of ritualized and exploratory routine use. Using these analytical accounts, we then decided on a smaller number (10) of illustrative episodes that we distinguished as those most clearly evidencing what we put forward in this paper as the fluid interplay between ritualized and exploratory engagement with assumption building. In the following section, we present our analysis of these episodes, sampling from the students’ work on the *Roadkill Rabbits* (4.1) and *Rabbits and Foxes* (4.2) problems.

4 Assumption building at the crossroads of mathematics and biology discourse during MM activity

In Viirman and Nardi (2017, 2018), we reported initial analyses tracing the evolution of the students’ mathematical discourse in two episodes, from *Yeast Growth* and *Digoxin*, in which there were hints of a scaffolding story about the gradual transition from ritualized to exploratory engagement with MM. We also saw signs of different forms of engagement in differing discourses, where the students were more competent, for instance, with forming and rejecting hypotheses about data when interpreting these data in biology terms than when treating the data on a purely mathematical level.

Here, we present two sets of episodes from the students’ work on *Roadkill Rabbits* in the first session and *Rabbits and Foxes* in the fourth session. In the first (4.1), we describe how the students’ engagement with assumption building routines mainly takes the form of ritual, through what we have labeled assumption building as “guesswork”. In the second (4.2), we highlight how the students draw on mathematical and biological discourses during assumption building. Through these episodes, we pay particular attention to evidencing the fluid interplay of ritualized and exploratory engagement in the students’ cross-disciplinary discursive activity.

4.1 Assumption building as “guesswork”: session 1, Roadkill Rabbits

The lecturer’s intended solution of *Roadkill Rabbits* (see also [Appendix](#)) was based on the observation that you can estimate the probability of a rabbit run over either as the fraction of the population per unit time being run over, or as the probability that a rabbit crossing the road is run over times the number of crossings an average rabbit makes per unit time. Combining

these two estimates, one can find the population density relatively easily. However, estimating the number of rabbit crossings per day and the probability of being hit while crossing requires making additional assumptions regarding the relationships between the variables. As previously mentioned, the principal aim of the problem was to engage the students in assumption building, not to come up necessarily with a complete solution.

Both groups² concluded that the crucial thing needed to solve the task was an estimate of the probability of a rabbit being hit by a car. However, instead of trying to formulate a model for this probability, using variables more easily estimated, they engaged in what we label as an *assumption building as “guesswork”* routine. Here is an excerpt from the early part of group B’s work on the task³:

A1: 56 cars over three hours, then you can find out how many cars that is per 24 h. And then that many cars killed that number of rabbits. And then you have to assume how many of the cars hit a rabbit. Is it, like, 5% of the cars that hit one, or is it all of them. It obviously isn’t, but it’s like, you have to assume something there.

B4: Mhm. Should you just say that it is, like, should you just guess?

A1: Yes, you have to assume something.

[20 sec pause]

A1: Because we don’t have all the information we need, in a way, so we need to assume ..., to make the best possible model.

B4: Yes, OK.

A1 made a number of implicit assumptions: the time it took to drive the 200 km (3 h); uniform traffic intensity, both throughout the day and in both directions; rabbits remain easily recognizable for 24 h. However, the students did not acknowledge them as such. Instead, the first explicit mention of the term “assumptions” came when A1 said that they “have to assume how many of the cars hit a rabbit”.⁴ This quantity differed from those previously mentioned, which could be estimated from facts given in the task, or from everyday experience. This was recognized by B4, who asked if they should “just guess”, to which A1 responded that “yes, you have to assume something”. This suggests that A1 considered assumption building as akin to “guesswork”—when you need the value of some quantity in order to solve the problem, you pick a number that seems reasonable. The students repeatedly engaged in this routine of assumption building as “guesswork”. However, they seemed concerned with the question of substantiation—see, for example, this excerpt, where the students were working on estimating the population density:

B2: Can’t we just divide the number of individuals by the area and then we find out how many there are per square meter?

A1: But we don’t know the area, we only know the length. We don’t know how wide it is.

B3: We’ll just have to make another assumption then, make it even more uncertain.

[Laughter]

² In the first session, there were two groups, A (students A2, A3, A4, C2) and B (students B1, B2, B3, B4 and A1).

³ Unless otherwise stated, all transcripts were translated into English by the first author.

⁴ In fact, it is the percentage of rabbits being hit by cars that they need, and they realize this soon after the excerpt presented here. However, this does not change the gist of our argument.

Again, we see the connection between assuming and guessing: introducing more assumptions implies greater uncertainty. The laughter seems to suggest that the students found the assumption building slightly ridiculous, but still they proceeded. We see this as indicative of a ritualized engagement with the assumption building routine: they did so possibly because they felt that it was expected of them, not because they thought it would help them construct a more plausible model. The repeated use of wording such as “have to” and “need to” is suggestive of this understanding on their part that they were somehow compelled to engage with assumption building as a core part of the problem-solving process.

Group A approached the problem slightly differently, but ran into similar difficulties. They spent the first few minutes discussing how to interpret the information given, and explicitly formulating assumptions based on this information. Their assumption building was thus closer to that described by the lecturer. However, having identified the percentage of rabbits being hit as the crucial factor, they too resorted to an *assumption building as “guesswork”* routine:

C2: But should we assume that all rabbits crossing the road are run over?

A3: Not when there are only 28 cars in 200 kilometers.

(...)

C2: It can't be one in three rabbits that is hit.

A4: No, probably closer to one in six or one in ten.

C2: Are you sure? How can you know that?

(...)

A4: They mate all the time, and they get, like, 10-12 at a time. So, there must be at least one in ten who dies.

C2: So, can we assume that one in ten dies then? Shall we do that?

Instead of trying to formulate relationships between the variables that might help them come up with an estimate, the students just picked a figure, albeit attempting to justify this pick drawing on narratives from biological discourse. They also explicitly connected assumption building with “guesswork”:

A2: I know. How is it possible to find an answer to this? You'll just have to assume, assume, assume. It's very difficult to find out.

C2: Yes, and you don't know what you need to assume either.

A2: No, we'll just have to guess, that's the thing.

C2: Yes.

The students appeared uncertain about what the assumption building they were expected to engage with was supposed to achieve. They saw this expectation as amounting to little more than guesswork. Again, we see this as a sign of ritualized engagement with assumption building routines. We also sense a frustration with the nature of the problem. This frustration is even more apparent in the following excerpt:

A3: This is a frustrating task. You think that it's a math task, but it turns out that you don't know anything, so you can't solve it.

C2: Yes, in a way we must find our own numbers.

A4: OK, if we assume that there are a hundred rabbits getting killed in two hours.

C2: It would have been much easier if he had just sat down and counted in one kilometer how many rabbits crossed the road.

We see this exchange as hinting at a commognitive conflict concerning the meaning of “math task”. For A3, the fact that “you don’t know anything, so you can’t solve it” disqualified it from being a mathematical problem, and C2 agreed, saying that “Yes, in a way we have to find our own numbers.” This suggests a view of a “math task” as something essentially closed, where the goals are clearly stated and all necessary information is provided. This contrasts with the lecturer’s description of modeling problems, as much more open and where it is necessary, for instance, to engage in assumption building routines that call for a creative take on interpreting the given information. This conflict also surfaces in group B, as in this comment by B4: “There are so many assumptions, in a way, you are used to doing exercises where the numbers should be this and this, done!”

There is another level to this commognitive conflict. In C2’s last aforementioned comment, he suggested an empirical approach. A1 in group B had a similar idea: “If the guy stops his car and observes a hundred metre stretch, how many rabbits cross the road in an hour and how many get run over, then you would know the percentage hit, right?” Besides showing that, when working within biological discourse, the students were capable of exploratory engagement with the problem (e.g., through suggesting alternative methods of solving the problem), this also hints at what they considered a valid solution to a modeling problem. For the lecturer, solving the problem meant using mathematical relationships between variables to construct a model. For the students, however, a meaningful solution to the problem involved collecting empirical data, while the mathematical approach was seen as merely “guesswork”. This conflict between biological/empirical and mathematical approaches recurred throughout the sessions, and we return to it in Sect. 4.2.

4.2 Translating assumptions into features of a mathematical model: session 4, Rabbits and Foxes

The lecturer’s intended solution of *Rabbits and Foxes* consists of interpreting five assumptions (assumptions 1–5, see Sect. 3.2 and Appendix) in mathematical terms, and then combining them to obtain two difference equations, one for the rabbits and one for the foxes. The students were able to engage more productively with this problem than with *Roadkill Rabbits* in that all three groups managed to produce at least parts of a mathematical model describing the development of the two species. In this sense, there was evidence that they could draw on a relatively familiar mathematical topic—they had worked with difference equations in previous sessions—in a relatively unfamiliar context—a system of difference equations they had not encountered before. Hence, we see this episode as an instance of a slight leap into less familiar mathematics than the episodes we reported in Sect. 4.1. At the same time, the students struggled with navigating between biological and mathematical discourse, a struggle that we see as signaling commognitive conflict. In what follows, we illustrate this struggle by looking more closely at the work of groups B and C.

Group B soon reached an insight into the interaction between the two species, summarized by B1 as follows: “The more rabbits there are, the more food for the foxes. And the more foxes, the fewer rabbits. And, then, there are fewer rabbits for the foxes to eat, and then they get fewer also.” However, the group was not making much progress beyond this. So, the researcher (R) intervened⁵:

⁵ As the researcher did not speak Norwegian, this and the next excerpt from group B were originally in English.

R: Let's read the assumptions. First, there is enough food for rabbits, and the population of rabbits increases, hm, if we only have that, what do you suppose will happen to the population?

B4: It would just grow and grow, and then it would reach the...

B3: Carrying capacity.⁶

R: A world full of rabbits.

B4: And then it would decrease.

R: My suggestion is now, look at the assumptions, work with each assumption, and then you can combine them, something like that.

[B4 quickly read Assumptions 2 and 3 aloud, but skipping the last part of Assumption 3, about the decrease being exponential.]

B4: But since we'll always have rabbits, because they have enough food, they won't die out. Can we think like that?

R: Yeah.

Here, the conflict between biological and mathematical discourse surfaced again. R tried engaging the students in mathematical discourse, looking at each assumption, and interpreting it in mathematical terms. We note that the majority of the assumptions were already partly mathematized, through the explicit mention of exponential growth and decay. The students, on the other hand, already had a sense of what the general behavior of the two species would be, based on their biological knowledge. Thus, they interpreted the assumptions within biological discourse, drawing on their knowledge of the real-world situation. In terms of the modeling process, R was already implicitly making use of the mathematization of the assumptions, whereas the students were still trying to make sense of the assumptions as such. For instance, although assumption 1 specified that food shortage was not a concern for the rabbits, B3 and B4 still spoke of the population reaching carrying capacity. The expected mathematization of assumption 1 incorporates the notion of unlimited growth, but in the students' biological discourse, this did not make sense. The researcher did not follow up on this, however. Instead, he concluded that the assumption of exponential growth would lead to a "world full of rabbits", an idea that is unrealistic from a biological point of view, and thus disregarded by the students. Furthermore, instead of viewing assumption 3 within mathematical discourse, as a description of what would happen to the quantity "foxes" in the absence of the quantity "rabbits", the students interpreted it as a biological statement, concluding that since the rabbits will not go extinct, neither will the foxes. Group C made a similar interpretation, treating the word "absence" not within mathematical discourse, but rather as something to be empirically checked:

C1: But can you use the same method on that one?

C1: Yes, but it doesn't grow exponentially, it grows in step with this one [the rabbits].

C1: It said here... Decreases. It decreases.

C2: In the absence of rabbits the population of foxes decreases.

C1: Absence, we don't know if there is an absence.

C2: At least there's not any absence yet.

Returning to the previous excerpt, R's affirmative response to B4's claim suggests that he was unaware of the commognitive conflict unfolding in these exchanges, and of how the different

⁶ Carrying capacity is a biological term indicating the population level for a species that can be supported by the environment in terms of food, living space, etc.

interpretations of the assumptions might lead to different conclusions. The researcher and the students were talking past one another, engaged in different discourses.

This conflict was also visible in the exchanges in group C. In the following excerpt, they were trying to make sense of the equations they had devised for the populations and had asked the researcher for advice. He had suggested that they go back to the assumptions, checking whether they have considered them all:

R: There is some information about how the foxes and rabbits interact. Is everything here [in the model], and does it make sense? What happens if there are zero rabbits, for instance? And what happens if there are zero foxes? And see if they [the equations] seem to do what you want them to do.

C2: Yes, zero foxes on...

C1: I'm thinking, if there is an epidemic, then that would have an effect.

C2: Yes.

C1: Like, if half of the foxes die.

C2: Yes.

C1: They haven't talked about that here.

C2: No, I think...

C1: If there is such a "fox plague", like there was last year...

Again, we see the conflict between mathematical and biological/empirical discourse. The researcher, already operating within the mathematized version of the problem, suggested a typical mathematical routine for checking the validity of a mathematical model, namely looking at extreme cases. The students, however, interpreted his statements empirically, considering possible reasons for a decline in foxes. The kind of "thought experiment" that the researcher suggests does not appear to be part of their discursive repertoire, at least not in this context. It might be the case that the students' slightly lesser familiarity with difference equations has driven them away from engagement with the mathematization the researcher is more comfortably engaged in. In any case, they seem to be more drawn to the empirical elements of the situation, and less to its mathematization.

Still, all three groups showed signs of exploratory engagement with the task, aiming at interpreting the assumptions and constructing narratives meaningful for solving the task. However, this exploratory discourse was again biological rather than mathematical, as in this excerpt, where group C was discussing assumption 2:

C2: It won't be $-F_n$, it won't be... Then they would eat one...

C1: No, that would be minus the number of foxes, and we don't know how many foxes that... well, then they would, like, eat one each.

C2: Yes.

C1: But, like, do fox cubs count?

C2: They eat too, but not...

C3: Maybe it's those that hunt rabbits themselves.

C1: They won't eat a whole on their own.

Here, the emphasis on the details of how the foxes' eating habits might influence the rabbit population stood in the way of the move towards mathematizing the problem and constructing a useful mathematical narrative. This difficulty in navigating between biological and mathematical discourse was also indicated by the absence of mathematical terminology in the students' talk. This was most evident in the exchanges in group B. Although the assumptions

explicitly mention exponential increase and decrease, the students avoided these terms in favor of broader terms such as “more”, “less”, “grow”, and “fewer”. This lack of a more precise description of the relationship between the populations made it difficult for the students to construct a model. Indeed, B4, drawing on this difficulty, made a more general comment on the challenges of constructing mathematical models: “Making a model, I think that’s really, really hard. Like, which number are you going to subtract, or plus, ehhm, multiply, I don’t really know. Is it an equation or...” We note how her talk about making the model mostly concerned what mathematical operations to perform. We see this process-orientation as a sign of ritualized engagement.

The following excerpt provides further evidence of how the students’ limited fluency with mathematical discourse leads them towards ritualized routine use. Group B was working on producing an expression for the difference between R_{n+1} and R_n , which would involve the growth rate k . The researcher asked how they would use the growth rate if they knew it:

B1: Just multiply it with k ?

R: k times what?

B2: Times R_n .

R: Exactly.

(...)

R: This is, you multiply the rate with the previous population. This is, you know, like, let’s say that this is the amount of the increase, not in total. Can you see my point?

B4: So we have to move it from..., to the other side?

(...)

B2: Oh, maybe like R_n equals.

R: R_{n+1} , you mean?

B2: Ah, yeah, R_{n+1} equals....

R: I always think that next year is what was the year before, plus something.

B4: Plus.

R: So this is the year before, yes.

B1: Plus R_n .

Here, we see signs of the students’ ritualized engagement with the algebraic routines needed to formulate the model. Where the researcher was trying to describe how one might reason mathematically about what may lie behind the equations and symbols, the students’ mathematical statements remained preoccupied with algebraic manipulations: “multiply it with k ”, “move it to the other side”, “plus R_n ”. Furthermore, these statements were largely formulated as questions, as appeals to the researcher as “ultimate substantiator” (Sfard, 2008, p. 234) for approval. We see this as a further sign of ritualized engagement.

5 Tracing the interplay between ritualized and exploratory engagement with MM activities in a biology context through a commognitive lens

The analyses from the study that we exemplify in this paper aimed to explore the following research questions: What evidence of ritualized and exploratory engagement with assumption building can be traced in biology students’ engagement with MM activities? What evidence of commognitive conflict can be traced in this engagement, especially in the students’ efforts to navigate across the discourses of mathematics and biology? In sum, in the course of the

students' engagement with the MM activities offered in these four sessions, there were clear indications of development in the students' engagement with assumption building. For instance, while, in the first session, their engagement with assumption building routines was ritualized as mainly involving "guesswork", in the fourth session, they were able to identify and formulate assumptions that underpinned their emerging mathematical models far more productively. That is, their engagement—in stark contrast to the first session—became gradually more confidently exploratory. At the same time, there were signs of persistent commognitive conflict in the students' activity. In resonance with Djepaxhija et al. (2015) and Galbraith and Stillman (2001), we observed what we labeled as an intra-mathematical conflict, concerning what is meant by a "math task" as well as an extra-mathematical, more far-reaching conflict between mathematical and biological/empirical discourse, concerning, for instance, what constitutes a plausible solution to a MM task, and how assumptions should be interpreted in a biological sense.

Furthermore, we saw clear signs of exploratory engagement with the tasks, but this engagement mainly took place within biological discourse, whereas the engagement with mathematical routines was mainly in the form of rituals, sometimes to the extent of interfering with the students' capacity to engage productively with the tasks. This is in line with the claims of Edelstein-Keshet (2005) and Madlung et al. (2011) that some mathematical experience—for example, familiarity with particular mathematical routines or prior experience with MM in school—is a prerequisite for benefiting more broadly from MM activities in the early phases of university studies. This also reflects our previous findings (Viirman & Nardi, 2017, 2018) where the students' ritualized mathematical routine use—for instance, their application of unsuitable graphing routines—impeded their exploratory engagement with the biological aspects of the tasks. In those papers, we hinted at how new routines may evolve, and particularly how discursants experience a step from ritualized to exploratory engagement. There is an inherent circularity in this process: a learner 'could not possibly appreciate the value of the new routine until she was aware of its advantages; such appreciation, however, could only emerge from its use' (Sfard, 2008, p. 246). In our present analyses, we see signs of this in how the assumption building routines were first seen as faintly ridiculous "guesswork", but, through their use, a deeper appreciation of their value and a more productive engagement began developing.

Such de-ritualization results in consolidated discourse, namely a 'well-developed network of interlacing, partially overlapping routines' (Sfard, 2008, p. 253–254). In this trajectory of growth, there are at least two conditions for effective mediation: the principle of the continuity of discourse ('introducing a new discourse by transforming an existing one', Sfard, 2008, p. 254) and the principle of commognitive conflict resolution. Our scrutiny of the dataset has led to the identification of evidence, or counterevidence, for both principles. For example, we identified evidence of the only partially resolved commognitive conflict experienced by the students around whether openly defined problems such as the ones they were given in these sessions can be treated as "math tasks". The discursants, students, and researchers present in the sessions appeared largely unaware of these conflicts. Indeed, although the project was well-received by the students (Viirman, Goodchild, & Rogovchenko, 2016) and had the full support of the funding institutions, we see an awareness of such potential conflicts as being of utmost importance if an integrated approach to MM in biology education is to be successful. The analyses we presented here indicate that engaging biology students in MM activities has definite potential, both as a motivational tool and to facilitate the de-ritualization of students' mathematical discursive activity.

We conclude with a discussion of the theoretical implications of our analyses. Earlier, we noted that students' reasoning is often described using binaries such as *instrumental* and *relational understanding* (Skemp, 1976) or *creative* and *imitative reasoning* (Lithner, 2008). Although perhaps not initially intended as such, binaries like these are often used as dichotomies, presupposing a strict divide between the two (Nardi, 2017). Indeed, this is sometimes the case also concerning the treatment of the commognitive constructs of ritual and exploration. However, in work concerning the notion of routine and presented in this issue (Lavie et al., 2018), ritual and exploration are viewed as the endpoints of a continuum, rather than as a dichotomy. By shedding light on the interplay between different types of student engagement with MM activities, in a context hitherto unexplored in commognitive research, we make a case for the potency, theoretical and pedagogical, of such a view, moving away from commonly used binaries and taking a more fluid perspective on the students' activity. For instance, the way the students in this study engaged in biological discourse around the assumptions in *Rabbits and Foxes* had elements of creativity that led us to characterize this engagement as exploratory. However, this creativity was hampered by ritualized (but not necessarily imitative) mathematical routine use. At the same time, the ritualized engagement with assumption building appears to have helped pave the way for more exploratory, creative engagement later. It is difficult to see how this complexity could be adequately described in terms of oppositional binaries. We believe that a more fluid view of the apparent dichotomy of the ritual-exploration dyad—where rituals morph into explorations through recognizing flexibility and broadened applicability—might help tap into the great interpretive potential that we, and others (e.g., Presmeg, 2016) see in the commognitive framework.

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Appendix: The lecturer's preferred solutions

Roadkill rabbits problem: The lecturer's preferred solution is based on the observation that you can estimate the probability of a rabbit getting run over in two ways. First, as the fraction of the population per unit time being run over. Second, as the product of the likelihood that a rabbit crossing the road is run over times the number of times an average rabbit crosses the road per unit time.

On the one hand, let R denote the size of the rabbit population, and K the number of rabbits killed daily on the highway. Then, the probability that a rabbit will get run over is $p = \frac{K}{R}$, where $\frac{K}{R}$ is the fraction of the population per unit time that gets run over. On the other hand, denote the average number of road crossings per unit time attempted by each rabbit by n , and the likelihood that a rabbit that attempts to cross the road will be run over by r . Then, $p = nr$. Combining these two equations, we arrive at $nr = \frac{K}{R}$, or equivalently $R = \frac{K}{nr}$. Assuming that the rabbits inhabit both sides of the highway along the entire 200 km stretch, we then get $d = \frac{R}{A}$, where d denotes the population density of rabbits inhabiting the area A along the road. To calculate the area A , we denote the width of the 200 km long stretch by w , so that $A = 200w$ km². Combining equations, we get $d = \frac{R}{A} = \frac{K}{200wnr}$ rabbits/km².

To find K (the number of rabbits killed each day), we need an estimate for the period during which 97 rabbits were run over. For simplicity, we assume that all 97 rabbits were killed in 1 day, and for ease of calculation, we round this off to $K = 100$ rabbits per day. However, the most difficult part remains: estimating n , that is, the number of road crossings per unit time attempted by an average rabbit. Suppose that the rabbits hop randomly about the area for 1 h/day. If they hop at a speed of s km/day, they cover about $\frac{s}{24}$ km/day, and, very roughly, will cross the road approximately $n = \frac{s}{24w}$ times/day. Substituting this into the previous equation, we get $d = \frac{100}{200w(\frac{s}{24w})r} = \frac{12}{sr}$ rabbits/km².

In order to estimate the likelihood r that a rabbit attempting to cross the road will be run over, we need to make additional assumptions, estimating the time it takes to cross the road and the time between passing vehicles. Supposing that the road is 5 m (0.005 km) wide, crossing takes $\frac{0.005}{s}$ days. Supposing that the average vehicle speed on the highway is 100 km/h = 2400 km/day, then the vehicle will travel $2400 \times \frac{0.005}{s} = \frac{12}{s}$ km while the rabbit is crossing, and if the vehicle is within that $\frac{12}{s}$ km stretch of the road at the time a rabbit begins its crossing, the rabbit will be killed. Next, we need to estimate the density of vehicles on the highway. We know that 28 vehicles passed by from the opposite direction in 2 h of travel. Thus, standing still by the highway for 2 h, we would see only half that many, or 14 vehicles, that is, 7 vehicles per hour. This means that the traffic density in one direction is 7 vehicles per 100 km or, equivalently, the vehicles are, on average, 14 km apart. Very roughly, we may estimate the probability r of a rabbit being run over as the ratio of the critical distance $12/s$ to the actual average interval, 14 km, or $r = \frac{12}{14s} \approx \frac{1}{s}$. Substituting this into the equation for d , we get $d = \frac{12}{s(1/s)} = 12$ rabbits/km².

Foxes and rabbits problem: Let R_n denote the population of rabbits and F_n the population of foxes at time n (in months). We now use the assumptions given to start constructing a model. Assumption 1 tells us that without foxes, the rabbit population would grow at a constant rate: $R_{n+1} - R_n = aR_n$, where a is the growth rate. Similarly, assumption 3 tells us that without rabbits, the foxes will decrease at a constant rate: $F_{n+1} - F_n = -cF_n$. Assumption 2 suggests that the presence of foxes decreases the growth rate of rabbits. This can be expressed in equation form in the following way: $R_{n+1} - R_n = aR_n - bR_nF_n$. Likewise, assumption 4 says that the presence of rabbits increases the rate of growth of foxes: $F_{n+1} - F_n = -cF_n + dR_nF_n$.

This model

$$R_{n+1} - R_n = aR_n - bR_nF_n$$

$$F_{n+1} - F_n = -cF_n + dR_nF_n$$

is often called the Lotka-Volterra model, named after two researchers who first devised it in the 1920s and 1930s. Note that in this, model b must be considerably smaller than a ; likewise, d should be considerably smaller than c .

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Affiliations

Olov Viirman^{1,2} • Elena Nardi³

¹ Department of Electronics, Mathematics and Natural Sciences, University of Gävle, Kungsbäcksvägen 47, 801 76 Gävle, Sweden

² MatRIC, University of Agder, Kristiansand, Norway

³ School of Education, University of East Anglia, Norwich, UK