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NORMATIVE SYSTEMS: CORE AND AMPLIFICATIONS

Lars Lindahl and Jan Odelstad

1. Introduction

1.1. Components of a normative system

By a normative system jurists often mean the law of a country, like Swedish law, or part of the law of a country, such as the Swedish law of contracts. Legal philosophers and logicians have raised the issue how a logical reconstruction of a normative system should be made. In this connection a central question is what kind of entities a normative system is composed of and how it should be represented.

In their well-known book *Normative Systems*¸ Carlos E. Alchourrón and Eugenio Bulygin conceive of a reconstructed normative system as a set of sentences deductively correlating pairs of sentences.[1] According to them, a set α of sentences deductively correlates a pair $\langle x,y\rangle$ of sentences if y is a deductive consequence of $\{x\}\cup\alpha$, or, in symbols, if $y\in\text{Cn}(\{x\}\cup\alpha)$. For α to be a normative system the additional requirement is made that there is at least one pair $\langle x,y\rangle$ where $y\in\text{Cn}(\{x\}\cup\alpha)$ such that x is a ”case” and y is a ”solution”. (A solution is a normative sentence expressed in terms of deontic operators for command, prohibition or permission). As observed by Alchourrón and Bulygin, the statement $y\in\text{Cn}(\{x\}\cup\alpha)$ is equivalent to $(x\supset y)\in\text{Cn}(\alpha)$ where $\supset$ is the symbol for truth-functional implication.

The approach in the present paper is similar to that of Alchourrón and Bulygin in important respects. We study normative systems essentially as deductive mechanisms yielding outputs for inputs. Also, we are interested in a rational reconstruction of normative systems, where the result is more logically elaborated than the original version. Finally, a central problem is that of finding a suitable formal framework for representing the logically elaborated version.

Our approach in the paper is explicitly algebraic. Implication is applied to conditions and is treated as a binary relation. Thus if a,b are two conditions, a binary relation ρ can be such that $a\rho b$ represents that a implies b. Normative

[1] Alchourrón & Bulygin (1971), pp. 54 f.

systems are represented in terms of relational structures where implicative relations are applied to a sets of conditions.

Treating conditions as objects provides a convenient way of introducing relations between conditions. Thus we may say, e.g. that according to some normative system, condition *a* is a ground for condition *b*. For example, it can be the case that, according to a particular regulation, not having a medical degree is a ground for not getting a license as a physician. Saying this is tantamount to saying that according to this regulation, a particular relation ρ holds between two conditions *a* and *b*. Or consider the statement: ”if *x* is a child of *y*, then *x* is entitled to inherit *y*.” Let us suppose that *a* is the binary condition ”to be a child of”, *b* is the binary condition ”to be entitled to inherit” and ρ represents the implicative relation. Then, in the example, we represent the original statement by $a\rho b$ (or $\langle a,b\rangle \in \rho$) without loss of information.[2]

Introducing relations between conditions naturally leads to an algebraic treatment of conditions and their relations, in the form of relational structures. In what follows, this idea will be developed by introducing what will be called Boolean quasi-orderings with domains of conditions.

In our approach, a normative system is represented by a class of implicative relational structures. One member of the class representing the normative system is the core structure, representing the uncontroversial and settled implicative contents of the system. The other members of the class are conceived of as ”amplifications” of the core. These amplifications represent different conceptions of the law, all of which conform with the core but which differ among themselves on issues which are not settled by the uncontroversial implicative contents.

Each member of the class of relational structures that represent the system (i.e., the class consisting of the core and its amplifications) only represents the ”positive” implicative statements of the normative system. Thus each relational structure represents statements of the kind ”α implies β”. Negative statements like ”not: α implies β”, in our approach, are taken into account as a general restriction on the entire class of relational structures that represent the system.

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[2] Conditions can be denoted either by expressions, using the sign of the infinitive, such as ”to be a woman”, ”to be a child of”, ”to be entitled to inherit”, or by expressions in the ing-form, like ”being a woman”, ”being a child of”, ”being entitled to inherit”.

1.2. Amplification and change

When the idea of core and amplifications is applied to an actual normative system, such as part of the law of a country, it is of great importance to distinguish between amplification and change. The case where the relational structure $\langle M,\rho_1\rangle$ represents an amplification of the core $\langle M,\rho\rangle$ of a normative system $\mathcal{S}$ should be kept apart from the case where $\langle M,\rho_1\rangle$ represents the core of a different, or ”changed”, normative system $\mathcal{S}_1$.[3]

Suppose that the following is an uncontroversial part of the legal system $\mathcal{S}$: Obtaining a gift by deceitful assertions implies being liable to having the gift annulled. Let a_1 be the antecedent and let b be the consequent in the example. Then $a_1\rho b$ is true for the core and for all amplifications. Now suppose that the question arises: Does as well obtaining a gift through careless misrepresentation (condition a_2) imply being liable to having the gift annulled? Suppose that this question is unsettled. This means that neither is $a_2\rho b$ true for all amplifications of the system, nor is there a restriction ”not $a_2\rho b$” on the system.

Two different developments of the example are to be distinguished. Firstly, we have the course of events where the law is changed by an act of legislation or by judicial precedent: the normative authority stipulates that a_2 implies b or it stipulates that not a_2 implies b. Then the original system $\mathcal{S}$ is replaced by a system $\mathcal{S}_1$ or by a system $\mathcal{S}_2$, respectively. If the enactment is to the effect that a_2 implies b, then we get a system $\mathcal{S}_1$ such that $a_2\rho_i b$ is true for all amplifications i of the system $\mathcal{S}_1$. If the enactment is to the effect that not a_2 implies b, then we get a system $\mathcal{S}_2$ where ”not $a_2\rho_j b$” is a restriction on all amplifications j of $\mathcal{S}_2$.

The course of events involving a ”change” in the sense now described should be distinguished from another course of events where the system $\mathcal{S}$ remains unaltered. Suppose that a subordinate court or authority, not having the power to change the law, pronounces that a_2 implies b or that not: a_2 implies b. Then the law is not changed. The statement $a_2\rho_i b$ is still not true for all amplifications i of $\mathcal{S}$, and there is still no restriction on all these models to the effect that not $a_2\rho_i b$.

It should be added that, in the example, the subordinate court or authority has not committed any judicial error if it chooses to act on an amplification, say ρ_k, such that $a_2\rho_k b$ is true for that amplification. But neither has the subordinate court or authority committed any judicial error if it rejects the amplification ρ_k.

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[3] In $\langle M,\rho\rangle$ and $\langle M,\rho_1\rangle$, M refers to a set of conditions and ρ, ρ_1 represent two implicative binary relations.

In either case the judge or official is not liable to being prosecuted for breach of duty.[4]

The assumption that, in the way just described, there can be opposite courses of action neither of which involves judicial error, does not imply that all of these courses of action are indifferent. With regard to a system $\mathcal{S}$, there can be good reasons for claiming that one amplification of $\mathcal{S}$ is better than another one, or that a particular restriction ought to be put on all amplifications of $\mathcal{S}$. The case that there are plausible claims of this nature, however, should be distinguished from the case that $\mathcal{S}$ is in fact replaced by a different system $\mathcal{S}_1$ or $\mathcal{S}_2$.[5]

2. Formal development

2.1. Conditions

Conditions can be of different arity (unary, binary etc.). Let us use $a,b,c,...,$ $a_1,b_1,c_1,...,$ $a_2,b_2,c_2,...$ for referring to conditions. Examples of conditions are: to be a woman, to be a parent of, of, to be a citizen of.

Sometimes we represent a condition by an expression $a(x_1,...x_\nu)$. (For example, the condition ”to be older than” then is represented by ”x is older than y”.) We note that, where, in this way, a condition is represented by an expression $a(x_1,...x_\nu)$, it is presupposed that $x_1,..,x_\nu$ are free variables which function as place-holders, and that $a(x_1,...x_\nu)$ is a sentence-form.

If a and b are ν-ary conditions, we form compound ν-ary conditions by $'$ (negation), $\wedge$ (conjunction), and $\vee$ (disjunction).[6] The negation of a condition a

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[4] If, in a law suit, the subordinate court or authority amplifies the law in a way that the losing party considers unjust, then (according the the Swedish Code of Procedure), within a certain time limit, this party can appeal against the judgment. The result of the appeal can, but need not, be that the original judgment is replaced. If, on the other hand, it is obvious that a subordinate court has ruled in a way contradicting the law, then, without time limit, the losing party can require a new trial. (In the latter situation, the Code of Procedure, Chapter 58, section 1:4, speaks of ”resning”, i.e. a new trial, in the case that ”the application of law, on which the judgment is based, manifestly is contrary to the law”.

[5] Some legal philosophers would claim that in well-developed legal systems there is always just one right answer to questions of law. This position can be expressed as the combination of two claims. One is the claim that among what we call amplifications there is always a best one. The other is the claim that the preference ordering among amplifications (or the set of principles which, when applied, yield this preference ordering) is itself a part of what should be called ”the law”. In this paper, we do not deal with the questions connected with these two claims.

[6] As is usual, the operation $\vee$ for disjunction is introduced as defined in terms of the operation $\wedge$ and $'$.

is defined by: for all $x_1,...,x_\nu$, $a'(x_1,...,x_\nu)$ iff not $a(x_1,...,x_\nu)$. The arity of a' is always the same as the arity of a. For example, since being a woman is a unary condition, not being a woman is unary as well. For conjunction the following rule is adopted. If a is μ-ary and b is ν-ary and ϕ = max$\{\mu,\nu\}$, then, for all $x_1,...,x_\phi$, $(a\wedge b)(x_1,...,x_\phi)$ iff $a(x_1,...,x_\mu)$ and $b(x_1,...,x_\nu)$. Thus the arity of $a\wedge b$ equals the greatest of the arities of a and b. For example, if a is the condition to ”be a woman” and b is the condition ”to be a parent of”, then $a\wedge b$ is the condition of being a mother of. As regards the identity relation = for conditions, if a is μ-ary and b is ν-ary and ϕ=max$\{\mu,\nu\}$, $a=b$ implies that, for all $x_1,...,x_\phi$, $a(x_1,...,x_\mu)$ iff $b(x_1,...,x_\nu)$.

The procedure of forming compounds can be iterated by the same principles. Thus, if $a\wedge b$ is being the mother of (see above) and c is being an adoptive parent of, $(a\wedge b)\vee c$ is the condition of being mother or adoptive parent of.

A condition a is *simple* if it is not compound. The ν-ary empty condition is the condition ? such that for no $x_1,...,x_\nu$, ?$(x_1,...,x_\nu)$. The ν-ary universal condition is the condition > such that for all $x_1,...,x_\nu$, >$(x_1,...,x_\nu)$. If $M=\{a,b,c,...\}$ is a set of conditions, then M^* is the set M closed under the operations $\wedge$ and $'$. The identity relation = for conditions used here is such that $\langle M^*,\wedge,'\rangle$ is a Boolean algebra, with = as its identity relation.

In the Boolean algebra $\langle M^*,\wedge,'\rangle$, ? is the zero element and > is the unit element. The arity of > and ? is maximal. Since, for any condition a, $a\wedge a'$ =? and $a\vee a'$=>, if the arity of ? (and of >) is ν, and the arity of condition a is μ, with $\mu\leq\nu$, then, for all $x_1,...,x_\nu$, $a\wedge a'(x_1,...,x_\mu)$ iff ?$(x_1,...,x_\nu)$, and $a\vee a'(x_1,...,x_\mu)$ iff >$(x_1,...,x_\nu)$.

Conditions have many affinities with relations, if, as is usual, relations are regarded extensionally as sets of ordered n-tuples. Obviously, the operations of negation, conjunction and disjunction for conditions have as counterparts the operations of complement, intersection and union for relations. However, if R_1 and R_2 are relations of different arity, their intersection $R_1\cap R_2$ is empty and their union is not a relation. For, example the intersection between a set of pairs and a set of triples is empty, and the union of a set of pairs and a set of triples is not a relation. As appears from the foregoing, the case is different with conditions.

Perhaps, it is appropriate to say that a theory of conditions in the way it is developed here is a modified theory of relations. In such a modified theory, sameness of arity is not presupposed for intersection and union. The treatment can be made algebraic, and relations between conditions can be introduced. In what follows, a small fragment of a modified theory of relations will be developed, in terms of conditions. [7]

2.2. Regular Boolean quasi-orderings, refinements and closures

In this section we give the main definitions and formal framework to be used for the rational reconstruction of a normative system.

DEFINITION. A structure $\langle B, \wedge, ', R\rangle$ is a *regular Boolean quasi-ordering* if $\langle B, \wedge, '\rangle$ is a Boolean algebra and R is quasi-ordering on B (i.e., a reflexive and transitive relation on B), which satisfies the following conditions for all a,b and c in B:

(1) cRa and cRb implies $cR(a\wedge b)$.
(2) aRb implies $b'Ra'$.
(3) $(a\wedge b)Ra$.
(4) not $\top R\bot$.

If $\langle B,\wedge,',R\rangle$ is a regular Boolean quasi-ordering and $\mathcal{B} = \langle B,\wedge,'\rangle$ we will often use $\mathcal{B}[R]$ to denote $\langle B,\wedge,',R\rangle$.[8] Let $\leq$ be the partial ordering determined by $\mathcal{B}$. From (3) it follows that $a\leq b$ implies aRb. For $a\leq b$ implies $a\wedge b=a$ and, since $a\wedge bRb$ it follows that aRb.

Any regular Boolean quasi-ordering determines a Boolean algebra in a natural way. To see this we use the following notation. Let Q be the indifference part of R. Then,

(1) $a_R=\{b\in B : bQa\}$.
(2) $B_R=\{a_R : a\in B\}$.
(3) $\leq_R$ is the relation on B_R defined by $a_R\leq_R b_R$ iff aRb.[9]

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[7] The authors have been inspired some ideas of Stig Kanger's. In the Fall of 1977, Kanger gave a series of lectures "New foundations of ethical theory" (not to be confounded with Kanger (1957), reprinted in Kanger (1971) which has the same title). The series was intended to deal with six principal subjects: conditions, causality and actions, deontics, preference theory, rights, justice. Unfortunately, only one part of the plan was realised, namely the part dealing with conditions. In this part, Kanger developed an algebraic theory of conditions, based on the work on Boolean algebras with operators by B. Jónsson and A. Tarski , and, in particular, the work on cylindric algebra by by L. Henkin, J.D. Monk, and A. Tarski (see Henkin, Monk, Tarski (1971)).

[8] The notions of Boolean quasi-ordering and regular Boolean quasi-ordering were introduced in Odelstad & Lindahl (1998). In the present essay, only regular Boolean quasi-orderings play a role.

[9] We note that $\leq_R$ is a well-defined partial ordering and that, since R is a quasi-ordering, Q is an equivalence relation.

If $\mathcal{B}[R]$ is a regular Boolean quasi-ordering, then $\mathcal{B} = \langle B_R, \leq_R\rangle$ is a Boolean algebra.[10] We say that $\mathcal{B}[R]$ *determines* the Boolean algebra $\mathcal{B}_R = \langle B_R, \leq_R\rangle$.

THEOREM 1. Suppose that $\{\mathcal{B}[R_i] | i\in I\}$ is a non-empty family of regular Boolean quasi-orderings. Then $\mathcal{B}[\cap_{i\in I}R_i]$ is a regular Boolean quasi-ordering.
Proof. (i) Let $\mathcal{B}[\mathsf{R}]=\mathcal{B}[\cap_{i\in I}R_i]$. We first prove that R is a quasi-ordering. Suppose that $x\mathsf{R}y$ and $y\mathsf{R}z$. Then xR_iy and yR_iz for all $i\in I$, which implies that xR_iz for all $i\in I$ and hence that $x\mathsf{R}z$. Thus R is transitive. Since xR_ix for all $i\in I$ it follows that $x\mathsf{R}x$. Therefore, R is reflexive. This shows that R is a quasi-ordering.
(ii) Suppose that $c\mathsf{R}a$ and $c\mathsf{R}b$. Then cR_ia and cR_ib for all $i\in I$. Since R_i is a regular Boolean quasi-ordering for all $i\in I$ it follows that $cR_i(a\wedge b)$ for all $i\in I$, which implies that $c\mathsf{R}(a\wedge b)$. In an analogous way we prove that $a\mathsf{R}b$ implies $b'\mathsf{R}a'$.
(iii) For all $i\in I$, $(a\wedge b)R_ib$, and therefore $(a\wedge b)\mathsf{R}b$. Furthermore, since, for all $i\in I$, not $\top R_i\bot$, it follows that not $\top\mathsf{R}\bot$. □

$\mathcal{B}[\rho]$ is a *supplemented* Boolean algebra if $\mathcal{B}$ is a Boolean algebra and ρ is a binary relation on B. A supplemented Boolean algebra $\mathcal{B}[\rho]$ is said to be *coherent* if there is a Boolean quasi-ordering $\mathcal{B}[R]$ such that $\rho\subseteq R$.

A regular Boolean quasi-ordering $\mathcal{B}[R]$ is:
(1) *a refinement* of the supplemented Boolean algebra $\mathcal{B}[\rho]$, if $\rho\subseteq R$;
(2) *a refinement* of the supplemented Boolean algebra $\mathcal{B}[\rho]$ *restricted* by σ, if $\rho\subseteq R$ and $R\subseteq\overline{\sigma}$.

The class of refinements of the supplemented Boolean algebra $\mathcal{B}[\rho]$ is denoted by $\mathsf{F}\mathcal{B}[\rho]$. Moreover, the class of refinements of $\mathcal{B}[\rho]$ restricted by σ is denoted by $\mathsf{F}_\sigma\mathcal{B}[\rho]$. Note that if $\mathcal{B}[R]$ is coherent, then $\mathsf{F}\mathcal{B}[\rho]$ is always non-empty, which does not hold for $\mathsf{F}_\sigma\mathcal{B}[\rho]$.

PROPOSITION 2.

(i) If $\rho_1\subseteq\rho_2$ and $\mathcal{B}[R]$ is a refinement of $\mathcal{B}[\rho_2]$, then $\mathcal{B}[R]$ is also a refinement of $\mathcal{B}[\rho_1]$.
(ii) If $\rho_1\subseteq\rho_2$, $\sigma_1\subseteq\sigma_2$ and $\mathcal{B}[R]$ is a refinement of $\mathcal{B}[\rho_2]$ with restriction σ_2, then $\mathcal{B}[R]$ is also a refinement of $\mathcal{B}[\rho_1]$ with restriction σ_1.
(iii) $\mathsf{F}_\varnothing\mathcal{B}[\rho]=\mathsf{F}\mathcal{B}[\rho]$.
(iv) If $\mathsf{F}_\sigma\mathcal{B}[\rho]\neq\varnothing$ then,
$\cap\{R \mid \mathcal{B}[R]\in\mathsf{F}_\sigma\mathcal{B}[\rho]\} = \cap\{R \mid \mathcal{B}[R]\in\mathsf{F}\mathcal{B}[\rho]\}$.

The *closure* of a coherent supplemented Boolean algebra $\mathcal{B}[\rho]$ is $\mathcal{B}[\mathsf{R}]$ where

[10] For a proof se Odelstad & Lindahl (1998), p.140.

$$\mathsf{R} = \cap\{R \mid \mathcal{B}[R] \in \mathsf{F}\mathcal{B}[\rho]\}.$$

The closure of $\mathcal{B}[\rho]$ is denoted by $\mathsf{C}\mathcal{B}[\rho]$.

PROPOSITION 3. If $\mathcal{B}[\rho]$ is a coherent supplemented Boolean algebra then

(i) $\mathsf{C}\mathcal{B}[\rho] \in \mathsf{F}\mathcal{B}[\rho]$.
(ii) If $\mathsf{C}\mathcal{B}[\rho] = \mathcal{B}[R]$, then $\rho \subseteq R$.
(iii) $\mathsf{C}(\mathsf{C}\mathcal{B}[\rho]) = \mathsf{C}\mathcal{B}[\rho]$.
(iv) If $\rho \subseteq \tau$ then $\mathsf{C}\mathcal{B}[\rho] \subseteq \mathsf{C}\mathcal{B}[\tau]$.

Note that (i) means that the closure of a supplemented Boolean algebra $\mathcal{B}[\rho]$ is a refinement of $\mathcal{B}[\rho]$, in fact the smallest such refinement. Statements (ii–iv) show that C satisfies the standard requirements for a closure operation, presupposing, of course, that the Boolean algebra is kept constant.[11]

Let us say that $\mathcal{B}[\rho|\sigma]$ is a *supplemented* Boolean algebra *with restriction* if $\mathcal{B}[\rho]$ is a supplemented Boolean algebra and σ is a binary relation such that $\rho \subseteq \overline{\sigma}$. By a *refinement* of $\mathcal{B}[\rho|\sigma]$ we mean a refinement of $\mathcal{B}[\rho]$ restricted with σ. We denote the class of refinements of $\mathcal{B}[\rho|\sigma]$ by $\mathsf{F}\mathcal{B}[\rho|\sigma]$. Thus the following holds:

$$\mathsf{F}\mathcal{B}[\rho|\sigma] = \mathsf{F}_\sigma\mathcal{B}[\rho].$$

A supplemented Boolean algebra with restriction $\mathcal{B}[\rho|\sigma]$ is said to be *coherent* if there is a Boolean quasiordering $\mathcal{B}[R]$ such that $\rho \subseteq R$ and $R \subseteq \overline{\sigma}$. Note that if $\mathcal{B}[\rho|\sigma]$ is coherent, then σ does not contain $\langle a,a\rangle$ for any element a in B, since R is reflexive, where $\mathcal{B}[R] \in \mathsf{F}\mathcal{B}[\rho|\sigma]$.

If $\mathcal{B}[\rho|\sigma]$ is a coherent supplemented Boolean algebra with restriction, then the *closure* of $\mathcal{B}[\rho|\sigma]$ is $\mathcal{B}[\mathsf{R}|\mathsf{S}]$ where

$$\mathsf{R} = \cap\{R \mid \mathcal{B}[R] \in \mathsf{F}\mathcal{B}[\rho|\sigma]\}$$

and

$$\mathsf{S} = \cap\{\overline{R} \mid \mathcal{B}[R] \in \mathsf{F}\mathcal{B}[\rho|\sigma]\}.$$

The closure of $\mathcal{B}[\rho|\sigma]$ is denoted by $\mathsf{C}\mathcal{B}[\rho|\sigma]$. Note that even if $\rho_1 \neq \rho_2$, it can still be the case that $\mathsf{C}\mathcal{B}[\rho_1] = \mathsf{C}\mathcal{B}[\rho_2]$, and that even if $\rho_1 \neq \rho_2$ and $\sigma_1 \neq \sigma_2$, it can still be the case that $\mathsf{C}\mathcal{B}[\rho_1,\sigma_1] = \mathsf{C}\mathcal{B}[\rho_2,\sigma_2]$.

PROPOSITION 4. Suppose that $\mathcal{B}[\rho|\sigma]$ is a coherent supplemented Boolean algebra with restriction. Then the following holds:

[11] See Birkhoff (1967), p. 111.

(i) If $\mathcal{B}[R] \in \mathsf{F}\mathcal{B}[\rho|\sigma]$ then $\sigma \subseteq \overline{R}$.
(ii) $\mathsf{C}\mathcal{B}[\rho|\sigma] \in \mathsf{F}\mathcal{B}[\rho|\sigma]$.
(iii) If $\mathsf{C}\mathcal{B}[\rho] = \mathcal{B}[\mathsf{P}]$ and $\mathsf{C}\mathcal{B}[\rho|\sigma] = \mathcal{B}[\mathsf{R}|\mathsf{S}]$ then $\mathsf{P} = \mathsf{R}$.
(iv) If $\mathsf{C}\mathcal{B}[\rho|\sigma] = \mathcal{B}[\mathsf{R}|\mathsf{S}]$ then $\mathsf{S} \subseteq \overline{\mathsf{R}}$.
(v) If $\mathsf{C}\mathcal{B}[\rho|\sigma] = \mathcal{B}[\mathsf{R}|\mathsf{S}]$ then $\rho \subseteq \mathsf{R}$ and $\sigma \subseteq \mathsf{S}$.
(vi) $\mathsf{C}(\mathsf{C}\mathcal{B}[\rho|\sigma]) = \mathsf{C}\mathcal{B}[\rho|\sigma]$.
(vii) If $\pi \subseteq \rho$ and $\sigma \subseteq \tau$ then $\mathsf{C}\mathcal{B}[\pi|\sigma] \subseteq \mathsf{C}\mathcal{B}[\rho|\tau]$.

The statements (v)–(vii) are a justification for calling C a closure operation, when it is applied to supplemented Boolean algebras with restrictions.

PROPOSITION 5. If $\mathsf{C}\mathcal{B}[\rho|\sigma] = \mathcal{B}[\mathsf{R}|\mathsf{S}]$, then for all *a,b* and *c* in *B*:
(i) $a\mathsf{S}b$ implies $a\mathsf{S}(b \wedge c)$.
(ii) $(a \wedge b)\mathsf{S}c$ implies $a\mathsf{S}c$.
(iii) $a\mathsf{S}b$ implies $b'\mathsf{S}a'$.
(iv) Not ?S>.

2.3. Rational reconstruction of a normative system

A normative system $\mathcal{S}$ is expressed in terms of a set of conditions *K*. $\mathcal{S}$ determines two relations ρ and σ on *K* according to the following rules:

$a\rho b$ iff $\mathcal{S}$ states that *a* implies *b*.
$a\sigma b$ iff $\mathcal{S}$ states that not: *a* implies *b*.

The conditions in *K* may be simple or compound. Let *L* be the set consisting of the simple conditions in *K* and of the simple conditions from which the compound conditions in *K* are constructed. Let *M* be the set *L* closed under the operations ∧ and ′, and let $\mathcal{M} = \langle M, \wedge, ' \rangle$.

The supplemented algebra $\mathcal{M}[\rho|\sigma]$ with restriction is the *crude condition representation* of $\mathcal{S}$. Suppose that $\mathcal{M}[\rho|\sigma]$ is coherent. Then the closure of $\mathcal{M}[\rho|\sigma]$ exists and we denote it $\mathcal{M}[\mathsf{R}|\mathsf{S}]$. We call $\mathcal{M}[\mathsf{R}|\mathsf{S}]$ *the elaborate condition representation* of $\mathcal{S}$, and we call $\mathcal{M}[\mathsf{R}]$ the *core* of $\mathcal{S}$. A refinement of $\mathcal{M}[\rho|\sigma]$ other than the core $\mathcal{M}[\mathsf{R}]$ is an *amplification* of $\mathcal{S}$. A *change* from $\mathcal{S}$ to a different system $\mathcal{S}_1$ implies that the elaborate condition representations of $\mathcal{S}$ and $\mathcal{S}_1$ are different.

Note that, although it holds that, if $a\sigma b$, then not $a\rho b$, it does not hold that if not $a\rho b$ then $a\sigma b$. $\mathcal{S}$ can for example be such that $(q_1 \wedge q_2 \wedge q_3)\rho r$ and $q_1\sigma r$ but neither $(q_1 \wedge q_2)\rho r$ nor $(q_1 \wedge q_2)\sigma r$. Also note that it follows from proposition 4(v) and 5(ii) that $\mathcal{S}$ cannot be such that $(q_1 \wedge q_2)\sigma r$ but not $q_1\sigma r$.

As an example showing part of the crude representation of a normative system, consider the question of citizenship according to the system of the U.S. Constitution.[12] In this system, the conjunction of the two conditions

a:.to be a person born or naturalised in the U. S.

b: to be a person subject to the jurisdiction of the U.S.

implies the condition

c: to be a citizen of the U.S.

This is represented in the form $(a\wedge b)\rho c$. On the other hand, in the same normative system, since it is a settled matter that citizens who are minors have not the right to vote in general elections, $a\wedge b$ does not imply the condition

d: to be entitled to vote in general elections.

Rather, part of the system is represented by $(a\wedge b)\sigma d$. (Note that $(a\wedge b)\sigma d$ is compatible with $(a\wedge b\wedge e)\rho d$, where e is an additional condition.)

Next, we illustrate that in different amplifications of the core of a normative system there can be different grounds for a particular condition. If two persons are not married, nevertheless they can have a relationship similar to being married. Suppose that the law does not specify exactly which conditions imply having a relationship similar to being married. However, there are a number of conditions that are criteria, such as cohabiting, having children in common, sharing economic assets and debts, having no legal impediments to marriage, etc. If all of these criteria are satisfied by persons x and y, they have a relationship ”similar to being married”. Conversely, if none of them is satisfied, they do not have such a relationship. However, the settled law does not say what is the result if some of the conditions are satisfied while others are not.[13]

Let us illustrate the performance of our framework by considering a highly simplified version of such an example. Suppose that $N=\{a,b,c\}^*$ and $\mathcal{N}=\langle N,\wedge,'\rangle$. Let $\mathcal{M}[\rho|\sigma]$, where $\rho=\{\langle a\wedge b,c\rangle\}$ and $\sigma=\{\langle a,b\rangle,\langle b,a\rangle,\langle a'\wedge b',c\rangle\}$, be a crude representation of a part of the normative system. Thus $\mathcal{M}[\rho|\sigma]$ is a supplemented Boolean algebra with restriction.

Now, it is possible to determine $\mathsf{F}\mathcal{M}[\rho|\sigma]$, i.e. the class of refinements of $\mathcal{M}[\rho]$ restricted by σ. Let us here only consider which elements in $\{a,b\}^*$ are related to c by refinements of $\mathcal{M}[\rho|\sigma]$. There are four possibilities:

(1) $a\wedge b$.

(2) $(a\wedge b)\vee(a\wedge b')$, i.e. a.

(3) $(a\wedge b)\vee(a'\wedge b)$, i.e. b.

[12] See The Constitution of the United States, Amendment XIV, section 1.

[13] For a more thorough discussion of this example, see Lindahl & Odelstad (1999), pp. 168 f., 175 f.

(4) $(a \wedge b) \vee (a \wedge b') \vee (a' \wedge b)$, i.e. $a \vee b$.

We see that in different amplifications there are different grounds for *c*. Note that it is not possible that a' is related to *c* by a refinement *R* of $\mathcal{M}[\rho|\sigma]$. For suppose that $a'Rc$. Since $(a' \wedge b')Ra'$ it follows by transitivity that $(a' \wedge b')Rc$. But since, moreover, $(a' \wedge b')\sigma c$ and $R \subseteq \overline{\sigma}$, we get a contradiction.

To illustrate the distinction between amplification and change, compare $\mathcal{M}[\rho|\sigma]$ with $\mathcal{M}[\rho|\sigma_1]$ where $\sigma_1 = \{\langle a,b \rangle, \langle b,a \rangle\}$. In contrast to what holds for $\mathcal{M}[\rho|\sigma]$, there are refinements of $\mathcal{M}[\rho|\sigma_1]$ such that a' is related to *c*.

2.4. A remark on concepts and couplings

The algebraic notions and results introduced in the present paper can be applied to elucidate the role of concepts within normative systems, in particular their role as ”couplings”. In some previous papers we have characterised different kinds of joinings of substructures of Boolean quasi-orderings (in particular, connections and couplings) and investigated their relationship. Since we cannot give the framework for that analysis in this essay, we must refer the reader to our principal papers on the subject.[14] It seems appropriate, however, by means of a highly simplified example, to give a hint of how our present framework can be combined with the theory of couplings.

Suppose that $\mathcal{M}[\rho|\sigma]$ is as in the example at the end of section 2.3. Let $A=\{a,b\}^*$ and $C=\{c\}$. Suppose that, for each *i*, $1 \leq i \leq 4$, where 1-4 relate to the possibilites (1)-(4) respectively, $\mathcal{M}[R_i]$ is a refinement of $\mathcal{M}[\rho|\sigma]$, and, thus, ρ_i is the implicative relation for the *i*-th possibility. Then $\langle a \wedge b,c \rangle$ is a coupling in $\mathcal{M}[R_1]$, $\langle a,c \rangle$ a coupling in $\mathcal{M}[R_2]$, $\langle b,c \rangle$ a coupling in $\mathcal{M}[R_3]$, and $\langle a \vee b,c \rangle$ a coupling in $\mathcal{M}[R_4]$. In our terminology, the coupling $\langle a \vee b,c \rangle$ is narrower than $\langle a,c \rangle$ and $\langle b,c \rangle$, and those two are narrower than $\langle a \wedge b,c \rangle$.[15] Generally it holds that a coupling in a refinement of a Boolean quasi-ordering $\mathcal{B}[R]$ is at least as narrow as the corresponding coupling in $\mathcal{B}[R]$.

3. Conclusion

In this paper we have used an algebraic approach to the problem of representing normative systems: such systems are treated as classes of Boolean quasi-orderings with domains consisting of conditions. For many logicians this kind of algebraic representation is perhaps not easy to digest. We will therefore emphasise that the algebraic notions introduced here might be used for studying

[14] See Lindahl & Odelstad (1999) and Odelstad & Lindahl (1998).

[15] See Lindahl & Odelstad (1999), pp. 175 f.

normative systems in a somewhat different way, namely if normative systems are represented as theories in the logical sense (i.e., as sets of sentences closed under logical consequence). To see this, suppose that T is a theory within predicate logic. If P and Q are ν-ary predicate symbols in the language of T, let $P\Im Q$ hold if and only if the sentence

$$\forall x_1,\ldots,x_\nu: P(x_1,\ldots,x_\nu) \rightarrow Q(x_1,\ldots,x_\nu)$$

is provable in T. The predicate symbols of T together with $\Im$ determines a Boolean quasi-ordering. This structure could be used for the study of certain conceptual aspects of T, for example how different subsets of predicates of T are coupled. The algebraic notions and results regarding connections and couplings in our previous work can be applied to the Boolean quasi-ordering and contribute to the clarification of the conceptual structure of T.

Another task for future work is the study of conceptual aspects of other kinds of systems than explicitly normative ones (as, for example legal systems). What we have in mind are, on the one hand, systems which are only implicitly normative, such as can be found in the social sciences or in everyday language, and on the other hand, theories and models in the natural sciences.

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References

Alchourrón, C.E. and Bulygin, E. (1971). *Normative Systems*. Springer-Verlag: Wien.

Birkhoff, G., (1967). *Lattice Theory*. 3rd ed. American Mathematical Society: Providence.

Henkin, L., Monk, J.D., and Tarski, A. (1971). *Cylindric Algebras.* Part 1. North-Holland: Amsterdam.

Kanger, S. (1957). *New Foundations for Ethical Theory*. Stockholm.

Kanger, S. (1971). ”New Foundations for Ethical Theory.” In R. Hilpinen, ed., *Deontic Logic: Introductory and Systematic Readings*. D. Reidel: Dordrecht. (A reprint of Kanger (1957).)

Lindahl, L. & Odelstad, J. (1999). ”Intermediate Concepts as Couplings of Conceptual Structures.” In: P. McNamara and H. Prakken, eds., *Norms, Logics and Information Systems.* IOS Press: Amsterdam.

Odelstad, J. & Lindahl, L. (1998). ”Conceptual Structures as Boolean Orderings.” In: L. Lindahl et al., eds., *Not Without Cause. Philosophical Essays Dedicated to Paul Needham on the Occasion of His Fiftieth Birthday*. Department of Philosophy: Uppsala.