CONCEPTUAL STRUCTURES AS BOOLEAN ORDERINGS

Jan Odelstad and Lars Lindahl

1.1. Preamble

The role played by concept formation in philosophy and science has been varying during our century. After some decades of rather low interest, there are signs indicating that the situation is changing. The aim of the present paper is to contribute to the study of this field. More specifically, our contribution aims at presenting a framework for the coupling of conceptual structures.

The first part of the paper presents the background and motivation of the framework to be introduced. The second part, which is the main part, introduces and develops the formal framework. Finally, in the conclusion, we add some remarks on future work to be done.[1]

1.2. Intermediate terms in law and ethics

Amendment XIV, section 1, of the Constitution of the United States reads as follows:

> All persons born or naturalized in the United States, and subject to the jurisdiction thereof, are citizens of the United States and of the State wherein they reside. No State shall make or enforce any law which shall abridge the privileges or immunities of citizens of the United States; nor shall any State deprive any person of life, liberty, or property, without due process of law; nor deny to any person within its jurisdiction the equal protection of the laws.

Two central terms in this constitutional rule are "citizen" and "person". The rule enumerates grounds for being a citizen of the United States and pronounces a number of legal consequences, expressed in terms of "shall", of this

[1] The authors are grateful to Sten Lindström and Sören Stenlund for valuable suggestions.

condition. It does not assert any grounds for being a person, but it pronounces a number of legal consequences attached to personhood.

Within the U.S. constitutional system, the article just referred to is supplemented by other rules established by the Constitution and by constitutional court decisions. These rules together, by specifying grounds and consequences, indicate the role of the term "citizen" or "person" within the system. For example, relating to the issue of the grounds for personhood, the question may arise whether a foetus is a person in the sense relevant for the U.S. Constitution. (Ronald Dworkin has forcefully argued that this is not the case, since it would imply the prohibition of abortion even in cases where, clearly, it is constitutionally permitted, for example when abortion is necessary in order to save a woman's life. [2]) And relating to the issue of consequences, the question may arise, for example, which kinds of discrimination between persons are incompatible with "equal protection of the law".

Let R be the term "person" (or "citizen") and let $F_1,...,F_m$ be terms used for indicating the grounds for personhood (citizenship), and $S_1,...,S_n$ be terms indicating the legal consequences. Then the role of R in the constitutional system can be illustrated by its occurrence in two arrays of rules, as follows.

For any $x_1,...,x_i$: If $F_1(x_1,...x_i)$, then $R(x_1,...,x_i)$
For any $x_1,...,x_i$: If $F_2(x_1,...,x_i)$, then $R(x_1,...,x_i)$
⋮
For any $x_1,...,x_i$: If $F_m(x_1,...,x_i)$, then $R(x_1,...,x_i)$

For any $x_1,...,x_i$: If $R(x_1,...,x_i)$, then $S_1(x_1,...,x_i)$
For any $x_1,...,x_i$: If $R(x_1,...,x_i)$, then $S_2(x_1,...,x_i)$
⋮
For any $x_1,...,x_i$: If $R(x_1,...,x_i)$, then $S_n(x_1,...,x_i)$

The arrays can be reformulated in two rules:

(1) For any $x_1,...,x_i$: If $F_1(x_1,...,x_i)$ or...or $F_m(x_1,...,x_i)$, then $R(x_1,...,x_i)$.
(2) For any $x_1,...,x_i$: If $R(x_1,...,x_i)$ then $S_1(x_1,...,x_i)$ and...and $S_n(x_1,...,x_i)$.

If the middle term R is eliminated, we get the single rule:

(3) For any $x_1,...,x_i$: If $F_1(x_1,...,x_i)$ or...or $F_m(x_1,...,x_i)$, then $S_1(x_1,...,x_i)$ and...and $S_n(x_1,...,x_i)$.

[2] Dworkin (1993), pp. 109 ff.

It is a well-known idea that in the formulation of legal rules a so-called "intermediate" term R such as "person" or "citizen" couples a set $S_1,\ldots,S_n$, indicating legal consequences, to a set $F_1,\ldots,F_m$, indicating legal grounds. In 1950, Alf Ross published his well-known essay on "Tû-Tû" dealing with this matter, but a debate in Scandinavian legal philosophy was started already in 1944-1945 by Anders Wedberg and Per Olof Ekelöf.[3] The paramount example of an intermediate term within this debate was the term "owner".

In the early Scandinavian debate, it was pointed out that intermediate terms can give economy of expression when rules are enacted by a legislator. The most economical way to express the rules of the two arrays above would seem to be by a single sentence like (3). This way out, however, would often be less convenient from the point of view of legislative technique, since there can be good reasons for placing different grounds and different consequences in different codes or statutes (private law, criminal law, taxation law etc.). From this point of view, a formulation in terms of the two sentences (1) and (2) would be undesirable as well. As remarked by Wedberg and Ross, however, if a formulation in terms of R and the two arrays of rules is chosen, some economy of expression will be achieved. For m grounds and n consequences, this formulation will yield a number of m plus n rules. If, in contrast, for each F_i and each S_j, a rule

(4) For any $x_1,\ldots,x_i$: If $F_i(x_1,\ldots,x_i)$, then $S_j(x_1,\ldots,x_i)$,

were enacted, we would have m times n rules instead.[4]

On the other hand, what is possibly a disadvantage connected with the use of intermediate terms in law and ethics, is their uncertain status as regards the distinction between descriptive and normative (or *Is* and *Ought*). For example, if a definition is sought of the term "citizen" as it is used within the U.S. constitutional system, is not obvious whether the definition should be formulated in terms of grounds or in terms of consequences, or in terms of both. A similar ambiguous status is often the case for terms used in moral and political argument. For example, it is not clear that the sentence "the action A is in the public interest" is wholly descriptive neither that it is wholly normative, since the concept "being in the public interest" seems to descriptive in part and normative in part.

[3] Ekelöf (1945), Ross (1951). Wedberg (1951), p. 246, refers to a lecture called "On the fundamental notions of jurisprudence", given by himself in the Law Club of Uppsala, as early as 1944. As mentioned by Wedberg, this lecture was based on a larger manuscript from that same year, on the logical analysis of legal science.

[4] Wedberg (1951), p. 273, and Ross (1957), p. 820.

Within the tradition from Hume and Bentham it is held to be of great importance to distinguish sharply between empirical statements on one hand, and statements that are normative or deontic on the other. As noted by Hume, when an author develops an argument, there is often an imperceptible shift from descriptive propositions in terms of *Is* and *Is not* to normative ones in terms of *Ought* and *Ought not*:

> In every system of morality, which I have hitherto met with, I have always remark'd, that the author proceeds for some time in the ordinary way of reasoning, and establishes the being of a God, or makes observations concerning human affairs; when of a sudden I am surpriz'd to find, that instead of the usual copulations of propositions, *is* and *is not*, I meet with no proposition that is not connected with an *ought* or *ought not*. This change is imperceptible; but is, however, of the last consequence.[5]

In legal argumentation, there exists a particular version of the kind of reasoning that Hume derides. This reasoning is often designated by the pejorative name "conceptual jurisprudence" or *Begriffsjurisprudenz*. Suppose that for some R the two arrays of rules relating $F_1,...F_m$ to R and R to $S_1,...,S_n$ are well established as parts of a legal system, but that it is controversial whether $S_1,...,S_n$ ought to be similarly related to F_{n+1}. "Conceptual jurisprudence" tries to find the answer to this question by deriving "$F_{n+1}(x_1,...,x_i)$ implies $R(x_1,...,x_i)$" from the meaning of R (or the concept expressed by R). The critics maintain that the argument is fallacious, since the meaning of R is only given by the two arrays, where F_{m+1} does not occur.[6] According to the critics, this fallacious argument only conceals the need for an independent argument that $S_1,...,S_n$ *ought* to be related to F_{n+1}. The need for an independent argument is emphasised by Hume:

> For as this *ought*, or *ought not*, expresses some new relation or affirmation, 'tis necessary that it should be observ'd and explain'd; and at the same time that a reason should be given, for what seems altogether inconceivable, how this new relation can be a deduction from others, which are entirely different from it.[7]

In the early Scandinavian debate, Wedberg and Ross both emphasised the function of intermediate terms as convenient technical devices, serving as vehicles of inference within a system of rules. Thus if we consider the

[5] Hume (1969), p. 521.

[6] See Lindahl (1985) p. 46.

[7] As is well-known, Bentham, and jurists influenced by him, maintain that the utilitarian principle provides a basis arguing for or against that $S_1,...S_n$ *ought* to be related to F_{n+1}. Hume (1969, p. 521), on the other hand, maintains that such a principle cannot be "perceived by reason": "the distinction of vice and virtue is not founded merely on the relations of objects, nor is perceiv'd by reason."

schematic formulations of rules in terms of the two arrays and sentences (1)-(3), all substantial information is given by (3), where *R* does not occur. The other formulations do not add anything substantial but are mere devices for expressing (3) in a more convenient way. (As we shall soon see, a similar idea has been maintained with regard to intermediate terms in natural science.)

1.3. Intermediate terms in probability theory and natural science

Outside law and morals, there are other fields where an intermediate term can serve as link between Is and Ought. One example is probability. Consider a statement of the kind "the probability of the event *A* equals *m*" (where *m* is a real number). Such a statement can be regarded as a link between statements relating to frequencies and symmetries, on the one hand, and statements about how one ought to choose between different games, on the other. It is a plausible idea that the so-called objective, or frequency, interpretation of probability deals with grounds for probability, whereas the so-called subjective interpretation deals with consequences, expressed in terms of Ought. This suggestion assigns a proper role to each of the two interpretations and may contribute to an improved understanding of probability theory.[8]

In natural science, the idea of "intermediate" has been applied to the term "force" within physical theory.[9] As is observed by Wedberg in his History of Philosophy, during the eighteenth century several thinkers thought of the forces spoken of in mechanics as a kind of mathematical fictions, useful for describing the movements of bodies in a convenient way.[10] What exists in physical reality, according to this view, are configurations of mass, speeds, and accelerations.

---

[8] Probability as a concept with grounds and consequences is discussed in Odelstad (1989).

[9] An interesting approach to the problem of intermediate terms in mechanics was outlined in the nineteenth century by Henri Poincaré. Poincaré pointed out that a proposition like (1) "the stars obey Newton's laws" can be broken up into two others, namely (2) "gravitation obeys Newton's laws" and (3) "gravitation is the only force acting on the stars". Among these, proposition (2) is a definition and not subject to the test of experiment, while (1) is subject to such a test. "Gravitation", according to Poincaré, is an intermediary. Poincaré maintains that in science, when there is a relation between two facts *A* and *B*, an intermediary *C* is often introduced by the formulation of one relationship between *A* and *C*, and another between *C* and *B*. The relation between *A* and *C*, then, is often elevated to a *principle*, not subject to revision, while the relation between *C* and *B* is a *law*, subject to such revision. See Poincaré (1905), pp. 124 f., in the chapter "Is science artificial?" (We are indebted to Sören Stenlund for this reference.)

[10] See Wedberg (1959), pp. 19 f. (1982), pp. 11 f.

Forces are fictions, but they enable us to describe the interrelations of the former entities in a compact way. As Wedberg mentions, Berkeley is among the thinkers who held this opinion.[11]

The position, held by Berkeley and others, that the use of the term "force" is merely a device for compact expression, closely resembles the idea of intermediate terms. This resemblance becomes even more obvious if the position in view is described in Wedberg's own words:

> If a body k with mass m is in a particular (spatial and temporal) relation to certain other bodies, we say that a force of magnitude f affects k. If a force of magnitude f affects k, then k receives an acceleration a satisfying the equation:
>
> (i) $f=a \cdot m$
>
> Thus the force occurs as a middle term in the pair of hypothetical statements:
>
> (ii) Given a certain configuration of mass, a certain force exists.
>
> (iii) Given a certain force, a certain acceleration results.
>
> If the middle term is eliminated, we arrive at the conclusion:
>
> (iv) Given a certain configuration of mass, a certain acceleration results.[12]

An objection to Berkeley's idea that forces are "fictions", however, is raised by Wedberg in pointing out that the term "force" can be defined in terms of such entities that Berkeley considers as real. Such a definition, in Wedberg's words, might be formulated as a definition of the entire statement:

> The body k exerts a force f upon the body k'.

A definition of this statement, then, can read as follows:

> f is the product of the acceleration a, which k' receives from k and the mass of k'.[13]

1.4. Frameworks for connecting conceptual structures

As described earlier, in legal systems and in ethics, intermediate terms are often used for coupling *Ought* and *Ought not* (or *Shall* and *Shall not*) to *Is* and *Is not*. In this case, the intermediate terms are used for coupling sentences of two different kinds, where it is problematic how sentences of one kind can be

---

[11] Wedberg (1959), p. 19, (1982), p. 12. refers to Berkeley's work *De Motu* from 1721. As Wedberg describes, Berkeley applied this idea not only to the term "force" but also to the terms "gravitation" and "attraction".

[12] Wedberg (1959), p. 19, (1982), p. 11. The idea of "force" as an intermediate term in the sense of the Scandinavian discussion is exploited as well in Götlind (1962).

[13] Wedberg (1959), p. 20, (1982), p. 12.

coupled to sentences of the other. As appears from the example concerning "force", this problem, i.e., of different kinds coupled, is not always present when there is a coupling by intermediate terms: "kinematic configuration" and "acceleration" are not of different kinds in a way that is philosophically problematic. Conversely, the connection or coupling between structures of different kinds can be philosophically problematic irrespective of whether intermediate terms are involved. Well-known examples, apart from the connection from descriptive to normative, are the connection from physical to mental and the connection from chemical to biological. At a very general level, in empirical science, there is the problem of the connection from observable to theoretical.

In some of the cases where the connection of different kinds is problematic (in particular as regards the connection from physical to mental) the notion of supervenience is used for clarifying the nature of the connection. Existing theories of supervenience, seem to us, however, to yield at best a very partial insight into the nature of the relation in view. In particular, they do not provide much information about the specific interrelations between parts of the two different structures.

It seems that an improved framework for defining the nature of connections and couplings is still to be achieved. If successful, such a framework would yield a better insight into the more comprehensive structure that encompasses both of the different but interrelated substructures. Obviously, the framework to be achieved should be appropriate as well for clarifying the role of intermediate terms.

2. Formal development

In a previous paper, we presented a first working model for defining the notions of "connection" and "coupling".[14] Intermediate concepts were in focus and viewed as representing couplings in the sense defined within the model. The present paper is different in several respects. The framework to be developed is based on the theory of Boolean algebra instead of lattice theory. The structures dealt with are not necessarily finite.[15] The basic kind of relations dealt with are quasi-orderings rather than partial orderings as was the case in our previous

[14] See Lindahl & Odelstad (1998). A first outline (of a preliminary character) was given in Lindahl & Odelstad (1996).

[15] In section 2.6, however, we present particular results for finite structures.

paper, where partial orderings were introduced by a transition to equivalence classes. The framework is abstract in the sense that the main results are not tied to a specific interpretation (in terms of conditions) as was the case in the earlier paper. Thus, the case where the domains of the orderings have conditions, or equivalence classes of conditions, as their members only plays the part of one of several models for the theory. Also, the new framework is more flexible in the sense that it is not confined to the joining of two subsystems within a background system. In the new framework it is possible to generate a background system when subsystems are given.

Although our theory is not confined to conditions structures, such structures still play an important part in our study of concepts and conceptual systems. Therefore, before the new framework is developed, we will give a short presentation of the notion of "implicational condition structure", which is central to our study of concepts.

2.1. Conditions and implicational condition structures

Obviously, a central notion in the study of implicational condition structures is that of a *condition.* [16] Conditions can be of different arity (unary, binary etc.).[17] In this subsection, let us use $a,b,c,...,$ $a_1,b_1,c_1,...,$ $a_2,b_2,c_2,...$ for referring to conditions. Examples of (binary) conditions are: to be the father of, to be the owner of, to be a citizen of, etc. Where a condition is represented by an expression $a(x_1,...x_\nu)$, we presuppose that $x_1,..,x_\nu$ are free variables which function as place-holders, and that $a(x_1,...x_\nu)$ is a sentence-form.

If a and b are ν-ary conditions, we form compound ν-ary conditions as follows. a' is the condition defined by $a'(x_1,...,x_\nu)$ iff not $a(x_1,...,x_\nu)$, $a\wedge b$ is defined by $(a\wedge b)(x_1,...,x_\nu)$ iff $a(x_1,...,x_\nu)$-and-$b(x_1,...,x_\nu)$, $a\vee b$ is defined by $(a\vee b)(x_1,...,x_\nu)$ iff $a(x_1,...,x_\nu)$-or-$b(x_1,...,x_\nu)$. Thus, we assume that $'$ is the operation for forming negations of conditions, $\wedge$ is the operation for forming conjunctions, and $\vee$ the operation for forming disjunctions. The procedure of forming compounds can be iterated. So, for example, $(a\wedge b)\vee c$ is a condition. A condition a is *simple* if it is not compound. The ν-ary empty condition is the

[16] For a more elaborate discussion, see Lindahl & Odelstad (1998).
[17] Note that we can always expand an m-ary condition c_m to an n-ary condition c_n by adding n-m dummy variables. Thus, if, in a particular context, the maximal arity of the conditions is ν, we can expand all the conditions to conditions of arity ν.

condition $\bot$ such that for no $x_1,...,x_\nu$, $\bot(x_1,...,x_\nu)$. The ν-ary universal condition is the condition $\top$ such that for all $x_1,...,x_\nu$, $\top(x_1,...,x_\nu)$.

If $M=\{a,b,c,...\}$ is a set of ν-ary conditions, then M^* is the set M closed under the operations $\wedge$ and $'$. Formally, M^* is defined as follows.

(1) If $a \in M$, then $a \in M^*$.
(2) If $a \in M$, then $a' \in M^*$.
(3) If $a,b \in B$, then $a \wedge b \in M^*$.
(4) The only members of M^* are those resulting from a finite number of applications of (1), (2) and (3).

For a, $b \in M^*$, $a = b$ expresses that a and b are extensionally equal due to logic alone, and the expression $a \vee b$ is defined by $a \vee b = a' \wedge b'$.

In those kinds of conceptual systems, which we call "implicational condition structures" there is a relation of implication between conditions. For example, in the conceptual system of the U. S. Constitution, the conjunction of the two conditions "to be a person born or naturalized in the U. S.", and "to be a person subject to the jurisdiction of the U.S." implies the condition "to be a citizen of the U.S.".

In our framework, we will use the letter R for a reflexive and transitive relation expressing the relation of implication within an implicational condition structure. Thus if a,b are ν-ary conditions, aRb means that for all $x_1,...,x_\nu$ it holds that if $a(x_1,...,x_\nu)$ then $b(x_1,...,x_\nu)$. For example, the particular statement just referred to in the system of the U.S. Constitution will be expressed by $p \wedge qRr$, where p,q,r are the specfic conditions mentioned above.

A structure $\langle M^*,R \rangle$, where M^* and R are as above, will be called an *implicational condition structure.*[18] In the next part of the paper, we introduce a number of formal properties of implicational condition structures. These properties are expressed as set-theoretical predicates. The notion of Boolean algebra and quasi-ordering will play an important role. As mentioned above, the framework introduced, however, is of a more general character and implicational condition structures are merely one among several kinds of models of the theory.

[18] In Lindahl &Odelstad (1988), several examples of implicational condition structures, taken from jurisprudence, are given and discussed.

2.2. *Quasi-orderings*

Let us first recall some standard notions in the theory of relations. The binary relation R on a set A is a quasi-ordering on A if R is transitive and reflexive. The strict part S of a quasi-ordering R is defined by aSb iff aRb and not bRa. The indifference part Q of a quasi-ordering R is defined by aQb iff aRb and bRa. (Trivially, since R is a quasi-ordering, Q is an equivalence relation.) If R is a quasi-ordering on the set A then $\langle A,R\rangle$ is a quasi-ordered set or a simply a quasi-ordering.[19]

Let $\langle A,R\rangle$ be a quasi-ordering and $X\subseteq A$. x is a least element in X relative to R if $x\in X$ and xRy for all $y\in X$. x is a greatest element in X relative to R if $x\in X$ and yRx for all $y\in X$. $a\in A$ is an upper bound for X if xRa for all $x\in X$. $a\in A$ is a lower bound for X if aRx for all $x\in X$. $a\in A$ is a least upper bound for X if a is an upper bound for X and aRb for all upper bounds b for X. $a\in A$ is a greatest lower bound for if a is a lower bound for X and bRa for all lower bounds b for X.

Note that a least element or a greatest element relative to a quasi-ordering $\langle A,R\rangle$ need not be unique, but if x and y are least elements (or greatest elements) in $X\subseteq A$, then xQy. The same holds for a least upper bound and a greatest lower bound. Let $lub_R X$ denote the set of all least upper bounds for X relative to R (in A). Let $glb_R X$ denote the set of all greatest lower bounds for X relative to R (in A). Thus $x,y\in lub_R X$ or $x,y\in glb_R X$ implies that xQy. If $x\in lub_R X$ and xQy then $y\in lub_R X$. Analogously for $glb_R X$. If R is a partial ordering, each of $lub_R X$ and $glb_R X$ consists of at most one element. If $lub_R X$ and $glb_R X$ are singletons, we denote the elements with $sup_R X$ and $inf_R X$ respectively.

We let $ub_R X$ denote the set of all upper bounds, and $lb_R X$ the set of all lower bounds, for X relative to R (in A). Also, we let $g_R X$ denote the set of greatest elements,and $l_R X$ the set of least elements, of X relative to R (in A). Note that $g_R(lb_R X)=glb_R X$ and $l_R(ub_R X)= lub_R X$.

In the following, we use "Proposition" to denote general and simple results that we do not prove.

PROPOSITION 1. If $\langle A,R\rangle$ is a quasi-ordering and $X_1\subseteq X_2\subseteq A$, then

(i) $ub_R X_1 \supseteq ub_R X_2$.

(ii) $lb_R X_1 \supseteq lb_R X_2$.

(iii) $c\in g_R X_1$ and $d\in g_R X_2$ implies cRd.

(iv) $c\in l_R X_1$ and $d\in l_R X_2$ implies dRc.

---

[19] Note that if R is a quasi-ordering and R is antisymmetric (i.e., if xQy implies $x=y$), then R is a partial ordering.

(v) $c \in lub_R X_1$ and $d \in lub_R X_2$ implies cRd.
(vi) $c \in glb_R X_1$ and $d \in glb_R X_2$ implies dRc.

PROPOSITION 2. Suppose that R_0 and R_1 are quasi-orderings on the set A, $X \subseteq A$, and $R_1 \subseteq R_0$.

(i) $l_{R_1} X \subseteq l_{R_0} X$.
(ii) $g_{R_1} X \subseteq g_{R_0} X$.
(iii) $lb_{R_1} X \subseteq lb_{R_0} X$.
(iv) $ub_{R_1} X \subseteq ub_{R_0} X$.
(v) $c_1 \in lub_{R_1} X$ & $c_0 \in lub_{R_0} X$ implies $c_0 R_0 c_1$.
(vi) $c_1 \in glb_{R_1} X$ & $c_0 \in glb_{R_0} X$ implies $c_1 R_0 c_0$.

PROPOSITION 3. Suppose that R_0 is a quasi-ordering on the set A_0, $A_1 \subseteq A_0$, and $R_1 = R_0/A_1$.[20] If $X \subseteq A_1$, then,

(i) $lb_{R_1} X \subseteq lb_{R_0} X$.
(ii) $ub_{R_1} X \subseteq ub_{R_0} X$.
(iii) $c_1 \in lub_{R_1} X$ & $c_0 \in lub_{R_0} X$ implies $c_0 R_0 c_1$.
(iv) $c_1 \in glb_{R_1} X$ & $c_0 \in glb_{R_0} X$ implies $c_1 R_0 c_0$.
(v) $A_1 \cap lub_{R_0} X \subseteq lub_{R_1} X$.
(vi) $A_1 \cap glb_{R_0} X \subseteq glb_{R_1} X$.[21]
(vii) $lb_{R_1} X \subseteq lb_{R_0} X$.
(viii) $ub_{R_1} X \subseteq ub_{R_0} X$.

If $\langle A,R \rangle$ is a quasi-ordering such that $lub_R\{a,b\} \neq \emptyset$ and $glb_R\{a,b\} \neq \emptyset$ for all $a,b \in A$, then $\langle A,R \rangle$ will be called a *quasi-lattice*. If $lub_R X \neq \emptyset$ and $glb_R X \neq \emptyset$ for all $X \subseteq A$, then a quasi-ordering $\langle A,R \rangle$ is a *complete quasi-lattice*.

2.3. Boolean quasi-orderings

A structure $\langle B, \wedge, ', R \rangle$ is a *Boolean quasi-ordering* if $\langle B, \wedge, ' \rangle$ is a Boolean algebra and R is quasi-ordering on B, which satisfies the following conditions:

---

[20] The expression R_0/A_1 denotes the restriction of the relation R_0 to A_1. Let R_0 be a ν-ary relation on A_0 where $A_1 \subseteq A_0$. Then, the restriction of R_0 to A_1 is $R_0 \cap A_1{}^{\nu}$.

[21] If R_0 is a quasi-ordering on A_0, then $R_1 = R_0/A_1$ is a quasi-ordering on A_1. Note that R_1 is not a quasi-ordering on A_0 if A_1 is a proper subset of A_0 (since R_1 is not reflexive on A_0.) With "$lub_{R_1} X$" we mean the set of least upperbounds for X with respect to R_1 as a quasi-ordering on A_1, and analogously with "$glb_{R_1} X$", "$lb_{R_1} X$", "$ub_{R_1} X$".

(1) cRa and cRb implies $cR(a\wedge b)$
(2) aRb implies $b'Ra'$.

If $\langle B, \wedge, ', R\rangle$ is a Boolean quasi-ordering and $\mathcal{B} = \langle B, \wedge, '\rangle$ we will often use $\mathcal{B}[R]$ to denote $\langle B, \wedge, ', R\rangle$.

Let $\leq$ be the partial ordering determined by the Boolean algebra $\langle B, \wedge, '\rangle$, in the sense that $a\leq b$ iff $a\wedge b=a$. Then $inf_{\leq}\{a,b\}=a\wedge b$. We use $\vee$ for the join-operation determined by $\langle B, \wedge, '\rangle$, i.e. $a\vee b$ is defined as $sup_{\leq}\{a,b\}$. Note that $a\vee b=(a'\wedge b')'$.

PROPOSITION 4. Suppose that $\mathcal{B}[R]$ is a Boolean quasi-ordering. Then,

(i) aRc and bRc implies $(a\vee b)Rc$,
(ii) $a\wedge b\in ub_R(lb_R\{a,b\})$.

Note that it does not hold generally in a Boolean quasi-ordering that $a\wedge bRa$ or $a\wedge bRb$. If this holds, then $a\wedge b\in lb_R\{a,b\}$ and, hence, $a\wedge b\in glb_R\{a,b\}$.

It is natural to say that $\mathcal{B}[R]$ is a *Boolean quasi-lattice* if $\mathcal{B}[R]$ is a Boolean quasi-ordering and $\langle B,R\rangle$ is a quasi-lattice. Furthermore, we say that a Boolean quasi-ordering $\mathcal{B}[R]$ is *regular* if not $\top R\bot$, and it holds that $a\leq b$ implies aRb. (Here $\bot$ is the zero element and $\top$ the unit element in $\langle B, \wedge, '\rangle$ and $\leq$ is the partial ordering determined by $\mathcal{B}$.) Note that if $\mathcal{B}$ is a Boolean algebra, then $\mathcal{B}[\leq]$ is a regular Boolean quasi-ordering (where $\leq$ is the partial ordering determined by $\mathcal{B}$).

LEMMA 5. Suppose that $\mathcal{B}[R]$ is a regular Boolean quasi-ordering. Then for all $a,b\in B$, $inf_{\leq}\{a,b\}\in glb_R\{a,b\}$ and $sup_{\leq}\{a,b\}\in lub_R\{a,b\}$.
Proof. Suppose that $a,b\in B$. Since $a\wedge b\leq a$, and $a\wedge b\leq b$, and $\langle B, \wedge, ', R\rangle$ is regular, it follows that $a\wedge bRa$ and $a\wedge bRb$. Hence, $a\wedge b$ is a lower bound for $\{a,b\}$ with respect to R. Suppose that cRa and cRb. Then, according to (1) in the definition of Boolean quasi-ordering, $cRa\wedge b$. Thus $(a\wedge b)\in glb_R\{a,b\}$. Since $(a\wedge b)=glb_{\leq}\{a,b\}$ the first part of the lemma follows. The second part of the lemma is proved analogously using proposition 4 (i). □

COROLLARY 6. If $\mathcal{B}[R]$ is a regular Boolean quasi-ordering, then $\langle B,R\rangle$ is a quasi-lattice.

We will now show that a regular Boolean quasi-ordering determines a Boolean algebra. We use the following notation:

(i) $a_R=\{b\in B : bQa\}$.[22]
(ii) $B_R=\{a_R : a\in B\}$.

[22] We recall that, since R is a quasiordering, Q is an equivalence relation.

(iii) $\leq_R$ is the relation on B_R defined by $a_R \leq_R b_R$ iff aRb. (We note that $\leq_R$ is a well-defined partial ordering.)

THEOREM 7. If $\mathcal{B}[R]$ is a regular Boolean quasi-ordering, then $\mathcal{B} = \langle B_R, \leq_R \rangle$ is a Boolean algebra.

Proof. We prove that $\mathcal{B} = \langle B_R, \leq_R \rangle$ is a distributed, complemented lattice. Within this proof, infimum and supremum is with respect to $\leq_R$.
(i) We first prove that $(a \wedge b)_R = inf\{a_R, b_R\}$. Since $\leq$ is a subset of R and since $a \wedge b \leq a$, $a \wedge b \leq b$, it follows that $(a \wedge b)Ra$ and $(a \wedge b)Rb$. Hence, $(a \wedge b)_R \leq_R a_R$ and $(a \wedge b)_R \leq_R b_R$, which implies that $(a \wedge b)_R$ is a lower bound of $\{a_R, b_R\}$ with respect to $\leq_R$. Let us now suppose that c_R is a lower bound of $\{a_R, b_R\}$, i.e. $c_R \leq_R a_R$ and $c_R \leq_R b_R$. Then, cRa and cRb, and from (1) it follows that $cR(a \wedge b)$, which implies $c_R \leq_R (a \wedge b)_R$. Hence, $(a \wedge b)_R = inf\{a_R, b_R\}$.
(ii) Let $\vee$ be the join operation defined by $\langle B, \wedge, ' \rangle$. The proof that $(a \vee b)_R = sup\{a_R, b_R\}$ is analogous with (i) above. $\langle B_R, \leq_R \rangle$ is thus a lattice.
(iii) Let us define two binary operations $\wedge_R$ and $\vee_R$ on B_R in the following way: $a_R \wedge_R b_R = inf\{a_R, b_R\}$, $a_R \vee_R b_R = sup\{a_R, b_R\}$. Then, since $(a \wedge b)_R = inf\{a_R, b_R\}$ and $(a \vee b)_R = sup\{a_R, b_R\}$, it follows that $a_R \wedge_R b_R = (a \wedge b)_R$ and $a_R \vee_R b_R = (a \vee b)_R$. The lattice $\langle B_R, \leq_R \rangle$ can equivalently be described as $\langle B_R, \wedge_R, \vee_R \rangle$.
(iv) Let $\top$ be the unit element and $\bot$ the zero element in $\langle B, \wedge, ' \rangle$. Then $\top_R$ is the greatest element and $\bot_R$ the least element in $\langle B_R, \leq_R \rangle$. Since not $\top R \bot$ it follows that $\bot_R \neq \top_R$. From $a \wedge a' = \bot$ it follows that $(a \wedge a')_R = \bot_R$ and hence $a_R \wedge_R a'_R = \bot_R$. In an analogous way we prove that $a_R \vee_R a'_R = \top_R$. For every element $x \in B_R$ there is an element $y \in B_R$ such that $x \wedge_R y = \bot_R$ and $x \vee_R y = \top_R$, which can be seen the following way. If $x = a_R$, then choose $y = a'_R$. Thus, $\langle B_R, \leq_R \rangle$ is a complemented lattice.
(v) We now prove that for all elements $x,y,z \in B_R$, $(x \vee_R y) \wedge_R z = (x \wedge_R z) \vee_R (x \wedge_R z)$. Suppose that $x = a_R$, $y = b_R$ and $z = c_R$. $(x \vee_R y) \wedge_R z = (a_R \vee_R b_R) \wedge_R c_R = (a \vee b)_R \wedge_R c_R = ((a \vee b) \wedge c)_R = ((a \wedge c) \vee (b \wedge c))_R = (a \wedge c)_R \vee_R (b \wedge c)_R = (a_R \wedge c_R) \vee_R (b_R \wedge c)_R = (x \wedge_R z) \vee_R (x \wedge_R z)$. In an analogous way we prove that that for all elements $x,y,z \in B_R$, $(x \wedge_R y) \vee_R z = (x \vee_R z) \wedge_R (x \vee_R z)$. Hence, $\langle B_R, \leq_R \rangle$ is a distributed lattice.
Thus, we have proved that $\langle B_R, \leq_R \rangle$ is a complemented, distributive, lattice, i.e., a Boolean algebra. □

We say that the regular Boolean quasi-ordering $\mathcal{B}[R]$ *determines* the Boolean algebra $\mathcal{B}_R = \langle B_R, \leq_R \rangle$.

2.4. Completeness and fragments

We start this section by introducing a convention that will be used in the remaining part of this paper. If $\mathcal{B}[R]$ is a Boolean quasi-ordering, the partial ordering which $\mathcal{B}$ determines will be denoted by $\leq$. In an analogous way, we use $\leq_i$ to denote the partial ordering determined by $\mathcal{B}_i$.

According to standard algebraic terminology, a lattice $\langle B,\wedge,\vee\rangle$ is *complete* if $inf_{\leq}A$ and $sup_{\leq}A$ exist for all $A\subseteq B$. A Boolean algebra $\langle B,\wedge,'\rangle$ is thus complete if $inf_{\leq}A$ (or $sup_{\leq}A$) exists for all subsets A of B. Generalizing this notion of completeness to quasi-orderings we say that a quasi-ordering $\langle B,R\rangle$ is complete if, for each $A\subseteq B$, $lub_RA\neq\varnothing$ and $glb_RA\neq\varnothing$.

For Boolean quasi-orderings we distinguish between four different kinds of completeness. A Boolean quasi-ordering $\mathcal{B}[R]$ is:

order complete if the quasi-ordering $\langle B,R\rangle$ is complete,

algebra complete if the Boolean algebra $\mathcal{B}$ is complete,

pre-complete if $\mathcal{B}[R]$ is algebra complete and for all $A\subseteq B$: if for all $a\in A$ it holds that cRa, then $cRinf_{\leq}A$,

complete if $\mathcal{B}[R]$ is order complete and pre-complete.

LEMMA 8. If the Boolean quasi-ordering $\mathcal{B}[R]$ is pre-complete, then for all $A\subseteq B$: if for all $a\in A$ it holds that if aRc, then $(sup_{\leq}A)Rc$.

Proof. Suppose that aRc for all $a\in A$. Since $\mathcal{B}[R]$ is a Boolean quasi-ordering it follows that $c'Ra'$ for all $a\in A$ and thus for all $a'\in A'$, where $A'=\{x'|\ x\in A\}$. From the pre-completeness of $\mathcal{B}[R]$ follows then that $c'Rinf_{\leq}A'$. Since $inf_{\leq}A'=(sup_{\leq}A)'$ according to De Morgan's Laws for infinite joins and meets of a Boolean algebra[23], it follows that $c'R(sup_{\leq}A)'$. Hence $(sup_{\leq}A)Rc$ since $\mathcal{B}[R]$ is a Boolean quasi-ordering. □

THEOREM 9. Suppose that $\mathcal{B}[R]$ is a regular Boolean quasi-ordering which is pre-complete. Then for all $A\subseteq B$, $inf_{\leq}A\in glb_RA$ and $sup_{\leq}A\in lub_RA$.

Proof. Let $A\subseteq B$. If $a\in A$ then $inf_{\leq}A\leq a$. Since R is regular it follows that $inf_{\leq}ARa$ for all $a\in A$. Hence, $inf_{\leq}A$ is a lower bound for A with respect to R. Suppose now that c is a lower bound for A with respect to R, i.e. if for all $a\in A$ it holds that if cRa. Then, since $\mathcal{B}[R]$ is pre-complete, $cRinf_{\leq}A$. Thus $inf_{\leq}A\in glb_RA$. The proof that $sup_{\leq}A\in lub_RA$ is done analogously using lemma 8. □

[23] See for example Sikorski (1960) p. 55.

COROLLARY 10. If a Boolean quasi-ordering is regular and pre-complete, then it is order complete.

We remind the reader of the definitions of a subalgebra, and of a complete subalgebra, of a Boolean algebra. If $\langle B, \wedge, '\rangle$ is a Boolean algebra and A is a non-empty subset of B such that A is closed under the operations $\wedge$ and $'$, then $\langle A, \wedge_A, '_A\rangle$ is a *subalgebra* of $\langle B, \wedge, '\rangle$ where $\wedge_A$ and $'_A$ are restrictions of the operations $\wedge$ and $'$ to A. (That A is closed under the operations $\wedge$ and $'$ means that if $a,b\in A$ then $a\wedge_A b \in A$ and $a'\in A$.) If $\langle A, \wedge_A, '_A\rangle$ is a subalgebra of $\langle B, \wedge, '\rangle$ we often omit the subscript A and denote it simply $\langle A, \wedge, '\rangle$. Suppose that $\langle A, \wedge, '\rangle$ is a subalgebra of $\langle B, \wedge, '\rangle$ and let $\leq$ be the partial ordering determined by $\langle B, \wedge, '\rangle$ and $\leq_A$ the partial ordering determined by $\langle A, \wedge, '\rangle$. Then $\leq_A = \leq/A$ and $inf_{\leq_A}\{a,b\} = inf_{\leq}\{a,b\}$ and $sup_{\leq_A}\{a,b\} = sup_{\leq}\{a,b\}$.

If $\mathcal{B}=\langle B, \wedge, '\rangle$ is a Boolean algebra, then $\mathcal{A}=\langle A, \wedge, '\rangle$ is called a *complete subalgebra* of $\mathcal{B}$ if $\mathcal{A}$ is a subalgebra of $\mathcal{B}$, and for all $X\subseteq A$, $inf_{\leq}X$ belongs to A whenever $inf_{\leq}X$ exists. [24] It follows from De Morgan's law (the infinite case) that $\mathcal{A}$ is a complete subalgebra of $\mathcal{B}$ iff for all $X\subseteq A$, $sup_{\leq}X$ belongs to A whenever it exists. Note that it holds generally that if $\langle B,\wedge,'\rangle$ is a Boolean algebra, and $\langle A,\wedge,'\rangle$ is a subalgebra of $\langle B,\wedge,'\rangle$, and $X\subseteq A$, then, $inf_{\leq}X\in A$ implies that $inf_{\leq}X = inf_{\leq_A}X$. Hence, if $\langle A,\wedge,'\rangle$ is a complete subalgebra of $\langle B,\wedge,'\rangle$, then $inf_{\leq}X = inf_{\leq_A}X$ whenever $inf_{\leq}X$ exists. Also, note that if $\mathcal{A}$ is a complete subalgebra of a complete Boolean algebra, then $\mathcal{A}$ is itself a complete Boolean algebra.

If $\mathcal{B}_0[R_0]$ is a Boolean quasi-ordering, and $\langle B_1,\wedge,'\rangle$ is a subalgebra of $\langle B_0,\wedge,'\rangle$, and $R_1=R_0/B_1$, then the structure $\langle B_1,\wedge,',R_1\rangle$ will be called a *fragment* of $\mathcal{B}_0[R_0]$.

THEOREM 11. If $\mathcal{B}_0[R_0]$ is a Boolean quasi-ordering and $\langle B_1,\wedge,',R_1\rangle$ is a fragment of $\mathcal{B}_0[R_0]$, then $\langle B_1,\wedge,',R_1\rangle$ is a Boolean quasi-ordering.

Proof. Since $\langle B_1,\wedge,'\rangle$ is a subalgebra of $\mathcal{B}_0$ it follows that $inf_{\leq_0}\{x,y\} = inf_{\leq_1}\{x,y\}$ for all $x,y\in B_1$. Suppose that $a,b,c\in B_1$ and cR_1a and cR_1b. Since $R_1=R_0/B_1$ it follows that cR_0a and cR_0b and hence $cR_0 inf_{\leq_0}\{a,b\}$. From this follows $cR_1 inf_{\leq_1}\{a,b\}$. This shows that $\langle B_1,\wedge,',R_1\rangle$ satisfies the condition (1) of a Boolean quasi-ordering.

Suppose that $a,b\in B_1$ and aR_1b. Then aR_0b, and from this follows that $b'R_0a'$. Since $\langle B_1,\wedge,'\rangle$ is a subalgebra of $\mathcal{B}_0$ it follows that $a',b'\in B_1$ and hence $b'R_1a'$. Thus $\langle B_1,\wedge,',R_1\rangle$ satisfies the condition (2) of a Boolean quasi-ordering. □

[24] See for example Sikorski (1960) p. 76.

If $\langle B_1,\wedge,',R_1\rangle$ is a fragment of a Boolean quasi-ordering, then since $\langle B_1,\wedge,',R_1\rangle$ is itself a Boolean quasi-ordering, it will be denoted by $\mathcal{B}_1[R_1]$.

We will now define three different kinds of fragments.

The Boolean quasi-ordering $\mathcal{B}_1[R_1]$ is called a *strong fragment* of $\mathcal{B}_0[R_0]$ if $\mathcal{B}_1$ is a complete subalgebra of $\mathcal{B}_0$ and $R_1=R_0/B_1$. Note that if $\mathcal{B}_1$ is a complete subalgebra of $\mathcal{B}_0$, and $X\subseteq B_1$, and $inf_{\leq_0}X$ exists, then $inf_{\leq_1}X=inf_{\leq_0}X$. And analogously, if $X\subseteq B_1$ and $sup_{\leq_0}X$ exists then $sup_{\leq_1}X=sup_{\leq_0}X$.

A fragment $\mathcal{B}_1[R_1]$ of $\mathcal{B}_0[R_0]$ is called *conservative* if for all $a,b\in B_1$ the following two conditions are satisfied:

(1) If $c\in glb_{R_0}\{a,b\}$ then there is $d\in glb_{R_1}\{a,b\}$ such that cQ_0d.

(2) If $c\in lub_{R_0}\{a,b\}$ then there is $d\in lub_{R_1}\{a,b\}$ such that cQ_0d.

A fragment $\mathcal{B}_1[R_1]$ of $\mathcal{B}_0[R_0]$ is called *strongly conservative* if $\mathcal{B}_1$ is a complete subalgebra of $\mathcal{B}_0$, and for all $A\subseteq B_1$ the following two conditions are satisfied:

(1) If $c\in glb_{R_0}A$ then there is $d\in glb_{R_1}A$ such that cQ_0d.

(2) If $c\in lub_{R_0}A$ then there is $d\in lub_{R_1}A$ such that cQ_0d.

LEMMA 12. If $\mathcal{B}_0[R_0]$ is a Boolean quasi-lattice and $\mathcal{B}_1[R_1]$ is a conservative fragment of $\mathcal{B}_0[R_0]$, then $\mathcal{B}_1[R_1]$ is a Boolean quasi-lattice.

Proof. From theorem 11 follows that $\mathcal{B}_1[R_1]$ is Boolean quasi-ordering. Suppose that $a,b\in B_1$. Since $\mathcal{B}_0[R_0]$ is a quasi-lattice, $glb_{R_0}\{a,b\}\neq\varnothing$. Let $c\in glb_{R_0}\{a,b\}$. From the assumption that $\mathcal{B}_1[R_1]$ is a conservative fragment of $\mathcal{B}_0[R_0]$ follows that there is $d\in glb_{R_1}\{a,b\}$ such that cQ_0d. Hence $glb_{R_1}\{a,b\}\neq\varnothing$. That $lub_{R_1}\{a,b\}\neq\varnothing$ is proved analogously. □

LEMMA 13. Suppose that $\mathcal{B}_0[R_0]$ is a Boolean quasi-ordering and $\mathcal{B}_1[R_1]$ is a strongly conservative fragment of $\mathcal{B}_0[R_0]$. Then the following holds:

(i) If $\mathcal{B}_0[R_0]$ is order complete, then $\mathcal{B}_1[R_1]$ is order complete and, for all $A\subseteq B_1$ it holds that $glb_{R_1}A = glb_{R_0}A\cap B_1$ and $lub_{R_1}A = lub_{R_0}A\cap B_1$.

(ii) If $\mathcal{B}_0[R_0]$ is pre-complete, then $\mathcal{B}_1[R_1]$ is pre-complete.

(iii) If $\mathcal{B}_0[R_0]$ is complete, then $\mathcal{B}_1[R_1]$ is complete.

Proof. (i) Suppose that $A\subseteq B_1$. Since $\mathcal{B}_0[R_0]$ is order complete $glb_{R_0}A\neq\varnothing$. Suppose that $c\in glb_{R_0}A$. According to condition (1) of a strongly conservative fragment there is $d\in glb_{R_1}A$ such that cQ_0d. Thus $glb_{R_1}A\neq\varnothing$. From condition (1) in the definition of a conservative fragment follows $glb_{R_1}A = glb_{R_0}A \cap B_1$. The remaining part of the proposition is proved analogously.

(ii) Suppose that $A \subseteq B_1$, and $c \in B_1$, and for all $a \in A$ it holds that cR_1a. Then cR_0a for all $a \in A$ and, since $\mathcal{B}_0[R_0]$ is pre-complete, $cR_0 inf_{\leq_0}A$, Since $\mathcal{B}_1$ is a complete subalgebra of $\mathcal{B}_0$, $inf_{\leq_1}A = inf_{\leq_0}$, and therefore $cR_0 inf_{\leq_1}A$. From $c \in B_1$ and $inf_{\leq_1}A \in B_1$ then follows that $cR_1 inf_{\leq_1}A$.

(iii) Follows immediately from (i) and (ii). □

LEMMA 14. If $\mathcal{B}_0[R_0]$ is a regular Boolean quasi-ordering and $\mathcal{B}_1[R_1]$ is a fragment of $\mathcal{B}_0[R_0]$, then $\mathcal{B}_1[R_1]$ is a regular Boolean quasi-ordering.

Proof. According to theorem 11, $\mathcal{B}_1[R_1]$ is a Boolean quasi-ordering. Note that $\leq_1 = \leq_0 / B_1$. Suppose that $a \leq_1 b$. Then $a \leq_0 b$, and since $\mathcal{B}_0[R_0]$ is regular, aR_0b. From this and $R_1 = R_0/B_1$ and $a,b \in B_1$ follows aR_0b. Since $\mathcal{B}_0[R_0]$ is regular follows that not $\top R_0 \bot$. Hence, not $\top R_1 \bot$. Thus, we have shown that $\mathcal{B}_1[R_1]$ is regular. □

THEOREM 15. If $\mathcal{B}_0[R_0]$ is a regular Boolean quasi-ordering and $\mathcal{B}_1[R_1]$ is a fragment of $\mathcal{B}_0[R_0]$, then $\mathcal{B}_1[R_1]$ is a conservative fragment of $\mathcal{B}_0[R_0]$.

Proof. Suppose that $c \in glb_{R_0}\{a,b\}$. Since $\mathcal{B}_0[R_0]$ is regular, it follows, according to lemma 5, that $inf_{\leq_0}\{a,b\} \in glb_{R_0}\{a,b\}$, and thus $cQ_0 inf_{\leq_0}\{a,b\}$. According to lemma 14, $\mathcal{B}_1[R_1]$ is regular, and hence $inf_{\leq_1}\{a,b\} \in glb_{R_1}\{a,b\}$. Since $\mathcal{B}_1$ is a subalgebra of $\mathcal{B}_0$, $inf_{\leq_1}\{a,b\} = inf_{\leq_0}\{a,b\}$. Hence, $cQ_0 inf_{\leq_1}\{a,b\}$. This shows that condition (1) of a conservative fragment is satisfied for $\mathcal{B}_1[R_1]$. That condition (2) is satisfied is proved analogously. □

THEOREM 16. If $\mathcal{B}_0[R_0]$ is a regular, pre-complete, Boolean quasi-ordering, and $\mathcal{B}_1[R_1]$ is a strong fragment of $\mathcal{B}_0[R_0]$, then $\mathcal{B}_1[R_1]$ is strongly conservative.

Proof. Suppose that $A \subseteq B_1$ and $c \in glb_{R_0} A$. From theorem 9 it follows that $inf_{\leq_0}A \in glb_{R_0}A$. Thus, $cQ_0 inf_{\leq_0}A$. The assumption that $\mathcal{B}_1[R_1]$ is a strong fragment of $\mathcal{B}_0[R_0]$, and therefore, $\mathcal{B}_1$ a complete subalgebra of $\mathcal{B}_0$, implies $inf_{\leq_0}A = inf_{\leq_1}A$. Thus, $cQ_0 inf_{\leq_1}A$. Since $inf_{\leq_1}A \in B_1 \cap glb_{R_0}A$ it follows from proposition 3 (vi) that $inf_{\leq_1}A \in glb_{R_1}A$. This shows that the condition (1) of a strongly conservative fragment is satisfied. That condition (2) is satisfied is proved analogously. □

2.5. Connections and couplings

Suppose that $\mathcal{B}_0[R_0]$ is a Boolean quasi-ordering and $\mathcal{B}_1[R_1]$ and $\mathcal{B}_2[R_2]$ are fragments of $\mathcal{B}_0[R_0]$. Then $\langle b_1,b_2\rangle$ is a *connection* from $\mathcal{B}_1[R_1]$ to $\mathcal{B}_2[R_2]$ in $\mathcal{B}_0[R_0]$ if the following four requirements are satisfied:

(i) $b_1 \in B_1$, $b_2 \in B_2$ and $b_1R_0b_2$

(ii) There is $a_1 \in B_1 \backslash B_2$ and $a_2 \in B_2 \backslash B_1$ such that $a_1R_0b_1$ and $b_2R_0a_2$

(iii) If $c \in B_1$ and $b_1R_0cR_0b_2$ then cQ_0b_1.

(iv) If $c \in B_2$ and $b_1R_0cR_0b_2$ then cQ_0b_2.

(ii)–(iii) are called the proximity principles.

A connection $\langle b_1,b_2\rangle$ from $\mathcal{B}_1[R_1]$ to $\mathcal{B}_2[R_2]$ in $\mathcal{B}_0[R_0]$ is called *strict* if $b_1S_0b_2$. Note that if $\langle b_1,b_2\rangle$ is strict then $b_1 \in B_1 \backslash B_2$ and $b_2 \in B_2 \backslash B_1$. A connection $\langle b_1,b_2\rangle$ from $\mathcal{B}_1[R_1]$ to $\mathcal{B}_2[R_2]$ in $\mathcal{B}_0[R_0]$ is called *indifferent* if $b_1Q_0b_2$.

THEOREM 17. Suppose that $\mathcal{B}_0[R_0]$ is a complete Boolean quasi-ordering and that $\mathcal{B}_1[R_1]$ and $\mathcal{B}_2[R_2]$ are strongly conservative fragments of $\mathcal{B}_0[R_0]$. If there is $a_1 \in B_1 \backslash B_2$ and $a_2 \in B_2 \backslash B_1$ such that $a_1S_0a_2$ then there is a connection $\langle b_1,b_2\rangle$ from $\mathcal{B}_1[R_1]$ to $\mathcal{B}_2[R_2]$ in $\mathcal{B}_0[R_0]$ such that $a_1R_0b_1$ and $b_2R_0a_2$.

Proof. (1) Suppose that $a_1 \in B_1 \backslash B_2$ and $a_2 \in B_2 \backslash B_1$ such that $a_1S_0a_2$. Then $a_1 \in \{c \in B_1 | cR_0a_2\}$. Since $\mathcal{B}_1[R_1]$ is order complete $\mathit{lub}_{R_1}\{c \in B_1|cR_0a_2\} \neq \varnothing$. Let $b_1 \in \mathit{lub}_{R_1}\{c \in B_1|cR_0a_2\}$. Hence, $b_1 \in B_1$ and $a_1R_0b_1$. Since $\mathcal{B}_0$ is order complete and $\mathcal{B}_1[R_1]$ is a strongly conservative fragment of $\mathcal{B}_0[R_0]$, $b_1 \in \mathit{lub}_{R_0}\{c \in B_1|cR_0a_2\}$. Since $a_2 \in \mathit{ub}_{R_0}\{c \in B_1|cR_0a_2\}$ it follows that $b_1R_0a_2$. Hence $a_2 \in \{c \in B_2|b_1R_0c\}$. $\mathcal{B}_2[R_2]$ is order complete and therefore $\mathit{glb}_{R_2}\{c \in B_2|b_1R_0c\} \neq \varnothing$. Let $b_2 \in \mathit{glb}_{R_2}\{c \in B_2|b_1R_0c\}$. Hence, $b_2 \in B_2$ and $b_2R_0a_2$. Since $\mathcal{B}_2$ is order complete and $\mathcal{B}_2[R_2]$ is a strongly conservative fragment of $\mathcal{B}_0[R_0]$, $b_2 \in \mathit{glb}_{R_0}\{c \in B_2|b_1R_0c\}$. Since $b_1 \in \mathit{lb}_{R_0}\{c \in B_2|b_1R_0c\}$ it follows that $b_1R_0b_2$. Thus, we have proved that $\langle b_1,b_2\rangle$ satisfies requirements (i) and (ii) of a connection.

(2) Suppose that $a \in B_1$ and $b_1R_0aR_0b_2$. From aR_0b_2 and $b_2R_0a_2$ follows that aR_0a_2. Therefore $a \in \{c \in B_1|cR_0a_2\}$ and thus, since $b_1 \in \mathit{lub}_{R_1}\{c \in B_1|cR_0a_2\}$, aR_0b_1. Hence aQ_0b_1, which proves that $\langle b_1,b_2\rangle$ satisfies requirement (iii) of a connection.

(3) Suppose that $a \in B_2$ and $b_1R_0aR_0b_2$. Thus $a \in \{c \in B_2|b_1R_0c\}$, which together with $b_2 \in \mathit{glb}_{R_0}\{c \in B_2|b_1R_0c\}$ implies that b_2R_0a. Hence, aQ_0b_2, which proves that $\langle b_1,b_2\rangle$ satisfies requirement (iv) of a connection. □

Suppose that $\mathcal{B}_0[R_0]$ is a Boolean quasi-ordering and $\mathcal{B}_1[R_1]$ and $\mathcal{B}_2[R_2]$ are fragments of $\mathcal{B}_0[R_0]$. Then $\langle b_1,b_2\rangle$ is a *coupling* from $\mathcal{B}_1[R_1]$ to $\mathcal{B}_2[R_2]$ in $\mathcal{B}_0[R_0]$ if $\langle b_1,b_2\rangle$ is a connection from $\mathcal{B}_1[R_1]$ to $\mathcal{B}_2[R_2]$ in $\mathcal{B}_0[R_0]$ and for all $a_1\in B_1$ and $a_2\in B_2$ the following holds: If $a_1R_0a_2$, then $a_1R_0b_1$ and $b_2R_0a_2$.

THEOREM 18. Suppose that $\mathcal{B}_0[R_0]$ is a Boolean quasi-ordering and $\mathcal{B}_1[R_1]$ and $\mathcal{B}_2[R_2]$ are fragments of $\mathcal{B}_0[R_0]$. If $\langle b_1,b_2\rangle$ is a coupling and $\langle b_3,b_4\rangle$ a connection from $\mathcal{B}_1[R_1]$ to $\mathcal{B}_2[R_2]$ in $\mathcal{B}_0[R_0]$ then $b_1Q_0b_3$ and $b_2Q_0b_4$.
Proof. Note that $b_1,b_3\in B_1$ and $b_2,b_4\in B_2$. From $b_3R_0b_4$ and the fact that $\langle b_1,b_2\rangle$ is a coupling from $\mathcal{B}_1[R_1]$ to $\mathcal{B}_2[R_2]$ in $\mathcal{B}_0[R_0]$ it follows that $b_3R_0b_1$ and $b_2R_0b_4$. Since $b_1R_0b_2$ we get $b_3R_0b_1R_0b_4$ and $b_3R_0b_2R_0b_4$. $\langle b_3,b_4\rangle$ is a connection, and from requirement (iii) and (iv) of a connection follows $b_1Q_0b_3$ and $b_2Q_0b_4$. □

2.6. Finite quasi-orderings

LEMMA 19. If $\mathcal{B}[R]$ is a finite Boolean quasi-ordering, then $\mathcal{B}[R]$ is pre-complete.
Proof. Since $\mathcal{B}$ is finite, it is complete. Let $\{a_1,...,a_m\}\subseteq B$ such that cRa_i, $1\leq i\leq m$. Note that $inf_{\leq}\{a_1,...,a_m\}=a_1\wedge...\wedge a_m$. Since $\mathcal{B}[R]$ is a Boolean quasi-ordering, $cRa_1\wedge a_2$. This, together with cRa_3 gives $cRa_1\wedge a_2\wedge a_3$, and so on. Finally, $cR\,inf_{\leq}\{a_1,...,a_m\}$. □

LEMMA 20. If $\mathcal{B}_0[R_0]$ is a finite Boolean quasi-ordering, and $\mathcal{B}_1[R_1]$ is a fragment of $\mathcal{B}_0[R_0]$, then $\mathcal{B}_1[R_1]$ is a strong fragment of $\mathcal{B}_0[R_0]$.
Proof. Let $\{a_1,...,a_m\}\subseteq B_1$. Since $inf_{\leq 0}\{a_1,...,a_m\}=a_1\wedge...\wedge a_m$, and since $\mathcal{B}_1$ is a subalgebra of $\mathcal{B}_0$, it results that $inf_{\leq 0}\{a_1,...,a_m\}\in B_1$. □

LEMMA 21. If $\mathcal{B}_0[R_0]$ is a finite, regular, Boolean quasi-ordering, then it is order complete.
Proof. This follows from lemma 19 and corollary 10. □

THEOREM 22. Suppose $\mathcal{B}_0[R_0]$ is a finite, regular, Boolean quasi-ordering, and that $\mathcal{B}_1[R_1]$ and $\mathcal{B}_2[R_2]$ are fragments of $\mathcal{B}_0[R_0]$. If there is $a_1\in B_1\setminus B_2$ and $a_2\in B_2\setminus B_1$ such that $a_1S_0a_2$, then there is a connection $\langle b_1,b_2\rangle$ from $\mathcal{B}_1[R_1]$ to $\mathcal{B}_2[R_2]$ in $\mathcal{B}_0[R_0]$ such that $a_1R_0b_1$ and $b_2R_0a_2$.
Proof. From Lemma 19 it follows that $\mathcal{B}_0[R_0]$ is pre-complete, and from lemma 20 it follows that $\mathcal{B}_1[R_1]$ and $\mathcal{B}_2[R_2]$ are strong fragments of $\mathcal{B}_0[R_0]$. Therefore, according to theorem 16, $\mathcal{B}_1[R_1]$ and $\mathcal{B}_2[R_2]$ are strongly conservative. From

lemma 21 it follows that $\mathcal{B}_0[R_0]$ is order complete. Thus, according to theorem 17, there is a connection $\langle b_1,b_2\rangle$ from $\mathcal{B}_1[R_1]$ to $\mathcal{B}_2[R_2]$ in $\mathcal{B}_0[R_0]$ such that $a_1R_0b_1$ and $b_2R_0a_2$. □

3. Conclusion

The notions of Boolean quasi-orderings and of fragments, as well as their different properties, developed in this paper, are intended to be means for studying implicational condition structures. Note that if $\langle M^*,R\rangle$ is an implicational condition structure, then $\langle M^*,\wedge,',R\rangle$ is a Boolean quasi-ordering. Many interesting conceptual structures can be represented as finite implicational condition structures, and then, theorem 22 can be applied for studying the links between different fragments of these structures.

The exposition in section 2 shows that the particular kinds of links between conceptual systems called "connection" and coupling" demand strong assumptions. It is plausible that there are several other kinds of links between such systems, only satisfying weaker requirements, which are worthy of attention and further study. One simple example might be a weaker kind of connection such that, presupposing that (ii)-(iv) in the definition of connection are fulfilled, $\langle b_1,b_2\rangle$ is a connection of this kind even if $b_1\in B_0\backslash B_1$ and $b_2\in B_0\backslash B_2$.

References

Dworkin, R., (1993). *Life's Dominion. An Argument about Abortion and Euthanasia.* HarperCollins, London.

Götlind, E. (1962). "Några kommenterar om "mening"." In A.-M. Henschen-Dahlquist (ed.) *Filosofiska Studier Tillägnade Konrad Marc-Wogau den 4 april 1962.* Uppsala.

Hume, D. (1969). *A Treatise of Human Nature.* First published 1739-40. Ed. with an introduction by E.C. Mossner. Penguin Books, London.

Lindahl, L. (1985). "Definitioner, begreppsanalys och mellanbegrepp i juridiken." In *Rationalitet och Empiri i Rättsvetenskapen.* Juridiska Fakultetens i Stockholm skriftserie, nr 6. Stockholm: Juridiska Fakulteten.

Lindahl, L. & Odelstad, J. (1996). "Grounds and consequences in conceptual systems." In S. Lindström et al. (eds.) *Odds and Ends. Philosophical Essays Dedicated to Wlodek Rabinowicz on the Occasion of His Fiftieth Birthday.* Uppsala Philosophical Studies 45. Department of Philosophy, Uppsala.

Lindahl, L. & Odelstad, J. (1998). "Intermediate concepts as couplings of conceptual structures." In H. Prakken & P. McNamara (eds.) *Norms, Logics and Informations Systems. New Studies on Deontic Logic and Computer Science*. IOS Press, Amsterdam-Oxford-Tokyo-Washington DC.

Odelstad, J. (1989). "Om den metodologiska subjektivismen." Lecture delivered in the higher seminar of theoretical philosophy, Uppsala University, April 1989. (Unpublished manuscript.)

Poincaré, H., (1907). *The Value of Science*. London. (English translation of *La Valeur des sciences*. Paris 1905.)

Ross, A. (1951). "Tû-Tû." In O.A. Borum & K. Illum (eds.) *Festskrift til Henry Ussing*. Juristforbundet, København.

Ross, A. (1956-57). "Tû-Tû", *Harvard Law Review* 70: 812-825. (English translation of Ross (1951)).

Sikorski, R., (1960). *Boolean Algebras*. Springer-Verlag, Berlin.

Wedberg, A. (1951). "Some problems in the logical analysis of legal science", *Theoria* 17: 246-275.

Wedberg, A. (1959). *Filosofins Historia. Nyare Tiden till Romantiken*. Bonniers, Stockholm.

Wedberg, A (1982). *A History of Philosophy*. Vol 2. The Modern Age to Romanticism. Clarendon Press, Oxford. (English translation of Wedberg (1959)).