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# Guaranteed a posteriori estimation of uncertain data in exterior Neumann problems for Helmholtz equation from inexact indirect observations of their solutions 

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#### Abstract

We consider the problem of guaranteed estimation of unknown right-hand sides of the equations entering the statement of the exterior Neumann problems for the Helmholtz equation from indirect observations of their solutions. A method is developed for the determination of guaranteed a posteriori estimates of this right-hand sides which are compatible with measurement data. It is shown that such estimates can be expressed via solutions of a uniquely solvable system of the Helmholtz equations.


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## 1. Introduction

Inverse problems associated with the analysis of exterior boundary value problems for the Helmholtz equation constitute an important part of the inverse scattering theory with major applications in electromagnetics and acoustics [1,2]. A review of results can be found in [2,3]. Obtaining guaranteed estimation of the unknown right-hand sides of equations and boundary conditions entering the statements of boundary value problems from observations of their solutions is a relatively new research direction. The settings and methods that simultaneously take into account the uncertainty of data for this class of inverse problems have been recently developed in [4,5].

In this paper, we extend the approaches set forth in our earlier studies [4,5] aimed at the analysis of inverse problems with uncertain data arising in the electromagnetic and acoustic diffraction theory. A goal is to establish a technique for obtaining guaranteed a posteriori estimates in electromagnetic and acoustic inverse problems formulated for the exterior Neumann problems for Helmholtz equations and provide its complete mathematical justification.

In order to determine such estimates, additional data (observations) are needed. Here we use indirect observations that are linear transformations of unknown solutions of the

[^0]exterior Neumann problems with additive deterministic errors. Such a kind of observation is motivated by the fact that unknown solutions often cannot be observed directly.

Assuming that unknown perturbations of right-hand sides of equations, Neumann boundary conditions, and errors in observations satisfy some quadratic restrictions, we define the set consisting of all right-hand sides entering into the exterior Neumann problem which are compatible with observation data. Any element belonging to this set is called an a posteriori estimate of unknown right-hand sides. Then we choose the optimal estimate among them, called the guaranteed a posteriori estimate.

As an optimality criterion, we use the guaranteed deviation between the a priori estimates of right-hand sides of the exterior Neumann problem and their exact values.

We show that the guaranteed a posteriori estimates of unknown right-hand sides are expressed via solutions of some uniquely solvable systems of Helmholtz equations.

## 2. Problem statement

Let $D$ be a bounded domain in $\mathbb{R}^{n}, n=2,3$, with the Lipschitz boundary $\Gamma$ having the unit outward normal $\nu, D^{\prime}=\mathbb{R}^{n} \backslash \bar{D}$, and $\varphi$ be a solution to the Neumann problem

$$
\begin{align*}
-\left(\Delta+k^{2}\right) \varphi & =f \quad \text { in } D^{\prime}  \tag{1}\\
\frac{\partial \varphi}{\partial v} & =g \quad \text { on } \Gamma  \tag{2}\\
\frac{\partial \varphi}{\partial r}-i k \varphi & =o\left(1 / r^{(n-1) / 2}\right), \quad r=|x|, r \rightarrow \infty \tag{3}
\end{align*}
$$

in which

- $k$ is a given nonzero complex number, $0 \leq \arg k<\pi$,
- $\partial \varphi / \partial \nu$ is a normal derivative of $\varphi$ on the boundary $\Gamma$,
- $f$ is a function from the space $\tilde{L}_{2}\left(D_{0}\right)$, where by $\tilde{L}_{2}\left(D_{0}\right)$ we denote the space of all complex-valued functions square-integrable in a bounded subdomain $D_{0} \subset D^{\prime}$ that vanish outside $\bar{D}_{0}$ in the domain $D^{\prime}$,
- $g$ is a function defined on $\Gamma$ and belonging to the space $L^{2}(\Gamma)$.

Setting (1)-(3) may be referred to [2,3] as the forward problems describing the scattering of a time-harmonic acoustic (or electromagnetic) wave of a (given) frequency $\omega$ by a rigid body $D, n=2,3$ (or a perfectly conducting cylinder having the cross-section $D \subset R^{2}, n=2$ ) with $\phi$ being the sound-pressure (or the polarized electromagnetic-field component) amplitude function; here $k=\omega / c$ is the wave number and $c$ is the sound (or light) speed in a homogeneous medium.

Denote by $\mathrm{d} \Gamma$ the element of measure on surface $\Gamma$, by $L^{2}(\Gamma)$ the space of squareintegrable functions on $\Gamma$, and by $H^{1}(\Omega)$ the standard Sobolev space of order 1 on the domain $\Omega$.

Introduce the spaces

$$
\begin{aligned}
H_{\mathrm{loc}}^{1}\left(D^{\prime}\right) & =\left\{u:\left.u\right|_{D^{\prime} \cap \Omega_{R}} \in H^{1}\left(D^{\prime} \cap \Omega_{R}\right) \text { for every } R>0 \text { such that } D^{\prime} \cap \Omega_{R} \neq \emptyset\right\}, \\
H_{\mathrm{loc}}^{1}\left(D^{\prime}, \Delta\right) & :=\left\{u: u \in H_{\mathrm{loc}}^{1}\left(D^{\prime}\right), \Delta u \in L_{\mathrm{loc}}^{2}\left(D^{\prime}\right)\right\},
\end{aligned}
$$

where

$$
L_{\mathrm{loc}}^{2}\left(D^{\prime}\right)=\left\{u:\left.u\right|_{D^{\prime} \cap \Omega_{R}} \in L^{2}\left(D^{\prime} \cap \Omega_{R}\right) \text { for every } R>0 \text { such that } D^{\prime} \cap \Omega_{R} \neq \emptyset\right\}
$$

$\Omega_{R}:=\{x:|x|<R\} ;$ the Laplacian is taken in the distributional sense.
It is proved that problem (1)-(3) has a unique solution such that $\varphi \in H_{\mathrm{loc}}^{1}\left(D^{\prime}, \Delta\right)$ (see, e.g. [6,7]).

Introduce the Hilbert space $\mathcal{H}:=\tilde{L}_{2}\left(D_{0}\right) \times L_{2}(\Gamma)$ with the inner product defined by

$$
\left(\mathbf{f}_{1}, \mathbf{f}_{2}\right)_{\mathcal{H}}:=\int_{D_{0}} f_{1}(x) \overline{f_{2}(x)} \mathrm{d} x+\int_{\Gamma} g_{1} \bar{g}_{2} \mathrm{~d} \Gamma \quad \forall \mathbf{f}_{1}=\left(f_{1}, g_{1}\right), \quad \mathbf{f}_{2}=\left(f_{2}, g_{2}\right) \in \mathcal{H} .
$$

Let $\Omega_{j}, j=1, \ldots, M$, be a given system of bounded subdomains of $D^{\prime}$ that do not intersect and have a piecewise smooth boundary. We suppose that one observes the functions of the form

$$
\begin{equation*}
y_{j}(x)=C_{j} \varphi(x)+v_{j}(x), \quad x \in \Omega_{j}, \quad j=1, \ldots, M \tag{4}
\end{equation*}
$$

where $v_{j}(x) \in L^{2}\left(\Omega_{j}\right)$ are observation errors and $C_{j} \in \mathcal{L}\left(L^{2}\left(\Omega_{j}\right), L^{2}\left(\Omega_{j}\right)\right)$ are linear continuous operators.

Suppose that there is the following a priori information about the data and errors: $\mathbf{f}:=$ $(f, g) \in \mathcal{H}$ and $v:=\left(v_{1}(\cdot), \ldots, v_{M}(\cdot)\right)$ belong to the set $G$ of the form

$$
\begin{aligned}
G= & \left\{(\mathbf{f}, v) \in H: \sum_{j=1}^{M} \int_{\Omega_{j}} D_{j}(x) v_{j}(x) \overline{v_{j}(x)} \mathrm{d} x+\int_{D_{0}} Q_{1}\left(f-f_{0}\right)(x) \overline{\left(f(x)-f_{0}(x)\right)} \mathrm{d} x\right. \\
& \left.\left.+\int_{\Gamma} Q_{2}\left(g-g_{0}\right) \overline{\left(g-g_{0}\right)} \mathrm{d} \Gamma\right) \leq \beta_{M}^{2}\right\}
\end{aligned}
$$

in which $\beta_{M}$ is a known number, $\mathbf{f}_{0}=\left(f_{0}, g_{0}\right) \in \mathcal{H}$ is a known vector-function, $D_{j}(x)$ are known measurable bounded positive continuous functions on $\bar{\Omega}_{k}$, and $Q_{1}$ and $Q_{2}$ are Hermitian positive definite operators in $L^{2}\left(D_{0}\right)$ and $L^{2}(\Gamma)$, respectively, for which there exist bounded inverse operators $Q_{1}^{-1}$ and $Q_{2}^{-1}$,

$$
H:=\tilde{L}_{2}\left(D_{0}\right) \times L_{2}(\Gamma) \times L^{2}\left(\Omega_{1}\right) \times \cdots \times L^{2}\left(\Omega_{M}\right) .
$$

Introduce the notion of a guaranteed a posteriori estimate of vector-function $\mathbf{f}=(f, g)$.
Definition 2.1: The set $G_{y}$ defined by

$$
G_{y}=\left\{\mathbf{f} \in \mathcal{H}: F(y, \mathbf{f}) \leq \beta_{M}^{2}\right\}
$$

is called the a posteriori set of all possible $\mathbf{f}=(f, g)$ corresponding to measurements (4) and $(\mathbf{f}, v)$ belonging to $G$, where $y:=\left(y_{1}(\cdot), \ldots, y_{M}(\cdot)\right)$,

$$
\begin{aligned}
F(y, \mathbf{f})= & \sum_{j=1}^{M} \int_{\Omega_{j}} D_{j}(x)\left(y_{j}(x)-C_{j} \varphi(x)\right) \overline{\left(y_{j}(x)-C_{j} \varphi(x)\right)} \mathrm{d} x \\
& +\int_{D_{0}} Q_{1}\left(f-f_{0}\right)(x) \overline{\left(f(x)-f_{0}(x)\right)} \mathrm{d} x+\int_{\Gamma} Q_{2}\left(g-g_{0}\right) \overline{\left(g-g_{0}\right)} \mathrm{d} \Gamma .
\end{aligned}
$$

Definition 2.2: The vector-function $\hat{\mathbf{f}}_{g}=\left(f_{g}, g_{g}\right)$ from the set $G_{y}$ is called a guaranteed a posteriori estimate of vector-function $\mathbf{f}=(f, g)$ if the following condition holds:

$$
\inf _{\mathbf{f}_{1} \in \mathbf{G}_{\mathbf{y}}} \sup _{\mathbf{f}_{2} \in \mathbf{G}_{\mathbf{y}}}\left\|\mathbf{f}_{1}-\mathbf{f}_{2}\right\|_{\mathcal{H}}=\sup _{\mathbf{f}_{\mathbf{2}} \in \mathbf{G}_{\mathbf{y}}}\left\|\hat{\mathbf{f}}_{g}-\mathbf{f}_{2}\right\|_{\mathcal{H}} .
$$

Definition 2.3: The quantity

$$
\delta_{a}=\sup _{\mathbf{f}_{2} \in G_{y}}\left\|\hat{\mathbf{f}}_{g}-\mathbf{f}_{2}\right\|_{\mathcal{H}}
$$

is called guaranteed error of a posteriori estimation.

Definition 2.4: A vector-function $\hat{\varphi}_{g}$ is called a guaranteed a posteriori estimate of unknown solution $\varphi$ if it uniquely solves problem (1)-(3) at $\mathbf{f}=\hat{\mathbf{f}}_{g}$.

Let us give an important remark. As a result of applying measurements $y_{j}(x), j=1, .$. , $m$, the set of unknown elements $\mathbf{f}=(f, g)$ becomes $G_{y}$. Functions $f(x), g(x)$ at which measurements $y_{j}(x), j=1, \ldots, m$, are given belong to this set. The elements of set $G_{y}$ are a posteriori estimates of unknown functions $f$ and $g$. Next, if $\mathbf{f}_{1}=\left(f_{1}, g_{1}\right)$ is a certain a posteriori estimate of the element $\mathbf{f}=(f, g)$, then the guaranteed error

$$
\delta\left(\mathbf{f}_{1}\right)=\sup _{\mathbf{f} \in G_{y}}\left\|\mathbf{f}_{1}-\mathbf{f}\right\|_{\mathcal{H}} .
$$

Note that this estimate is not worse than the a priori estimate $\left(f_{0}, g_{0}\right)=\mathbf{f}_{0}$ because from the proof of Theorem 3.1 it follows that $\delta\left(\mathbf{f}_{g}\right)=\inf _{\mathbf{f}_{1} \in \mathcal{H}} \delta\left(\mathbf{f}_{1}\right)$, and therefore the inequality $\delta\left(\mathbf{f}_{0}\right) \geq \delta\left(\mathbf{f}_{g}\right)$, holds.

## 3. Main results

In order to obtain the representation for a posteriori estimates, we first prove the following assertion.

Lemma 3.1: There exists a unique element $\hat{\mathbf{f}}=(\hat{f}, \hat{g}) \in \mathcal{H}$ such that

$$
\inf _{\mathbf{f} \in \mathcal{H}} F(y, \mathbf{f})=F(y, \hat{\mathbf{f}})
$$

which is determined by

$$
\begin{equation*}
\hat{f}(x)=\left.\chi_{D_{0}}(x) Q_{1}^{-1} \hat{p}(x)\right|_{D_{0}}+f_{0}(x), \quad \hat{g}=\left.Q_{2}^{-1} \hat{p}\right|_{\Gamma}+g_{0} \tag{5}
\end{equation*}
$$

where $\hat{p} \in H_{\mathrm{loc}}^{1}\left(D^{\prime}, \Delta\right)$ is uniquely defined from the solution of the problem

$$
\begin{align*}
-\left(\Delta+\bar{k}^{2}\right) \hat{p}(x) & =-\sum_{j=1}^{M} \chi_{\Omega_{j}}(x) C_{j}^{*} D_{j}\left[C_{j} \hat{\varphi}-y_{j}\right](x) \quad \text { in } D^{\prime},  \tag{6}\\
\frac{\partial \hat{p}}{\partial v} & =0 \quad \text { on } \Gamma,  \tag{7}\\
\frac{\partial \hat{p}}{\partial r}+i \bar{k} \hat{p} & =o\left(1 / r^{(n-1) / 2}\right), \quad r=|x|, \quad r \rightarrow \infty  \tag{8}\\
-\left(\Delta+k^{2}\right) \hat{\varphi}(x) & =\left.\chi_{D_{0}}(x) Q_{1}^{-1} \hat{p}(x)\right|_{D_{0}}+f_{0}(x) \quad \text { in } \quad D^{\prime},  \tag{9}\\
\frac{\partial \hat{\varphi}}{\partial v} & =Q_{2}^{-1} \hat{p}+g_{0} \quad \text { on } \Gamma  \tag{10}\\
\frac{\partial \hat{\varphi}}{\partial r}-i k \hat{\varphi} & =o\left(1 / r^{(n-1) / 2}\right), \quad r=|x|, \quad r \rightarrow \infty \tag{11}
\end{align*}
$$

Here, $\hat{\varphi} \in H_{\mathrm{loc}}^{1}\left(D^{\prime}, \Delta\right), \chi_{M}(x)=\left\{\begin{array}{ll}1, & x \in M \\ 0, & x \notin M\end{array}\right.$ is a characteristic function of the set $M \subset \mathbb{R}^{n}$.
Functional $F(y, \mathbf{f})$ can be represented in the form

$$
\begin{equation*}
F(y, \mathbf{f})=F(y, \hat{\mathbf{f}})+F_{1}(\mathbf{f}-\hat{\mathbf{f}}) \tag{12}
\end{equation*}
$$

where

$$
\begin{equation*}
F_{1}(\mathbf{f})=\sum_{j=1}^{M} \int_{\Omega_{j}} D_{j}(x) C_{j} \varphi(x) \overline{C_{j} \varphi(x)} \mathrm{d} x+\int_{D_{0}} Q_{1} f(x) \overline{f(x)} \mathrm{d} x+\int_{\Gamma} Q_{2} g \bar{g} \mathrm{~d} \Gamma \tag{13}
\end{equation*}
$$

Proof: It is easy to see that functional $F(y, \mathbf{f})$ can be decomposed as

$$
F(y, \mathbf{f})=F_{1}(\mathbf{f})+L(y, \mathbf{f})+C_{0}(y)
$$

where $F_{1}(\mathbf{f})$ is defined by (13),

$$
\begin{aligned}
L(y, \mathbf{f}) & =-2 \Re\left(\sum_{j=1}^{M}\left(D_{j} C_{j} \varphi, y_{j}\right)_{L^{2}\left(\Omega_{j}\right)}+\left(Q_{1} f, f_{0}\right)_{L^{2}\left(D_{0}\right)}+\left(Q_{2} g, g_{0}\right)_{L^{2}(\Gamma)}\right) \\
C_{0}(y) & =\sum_{j=1}^{M} \int_{\Omega_{j}} D_{j}(x) y_{j}(x) \overline{y_{j}(x)} \mathrm{d} x+\int_{D_{0}} Q_{1} f_{0}(x) \overline{f_{0}(x)} \mathrm{d} x+\int_{\Gamma} Q_{2} g_{0} \overline{g_{0}} \mathrm{~d} \Gamma .
\end{aligned}
$$

Using the fact that the solution $\varphi(x)$ to problem (1)-(3) can be represented as

$$
\varphi(x)=C \mathbf{f}(x):=\int_{D_{0}} \Phi_{k}(x, y) f(y) \mathrm{d} y+\int_{\Gamma} \Phi_{k}(x, y) g(y) \mathrm{d} \Gamma_{y} \quad \text { in } D^{\prime}
$$

where $\Phi_{k}(x, y)$ is the Green function satisfying the condition ${ }^{1} \partial \Phi_{k}(x, y) / \partial v_{x}=0$ on $\Gamma$, and introducing the operators $A_{j}: \mathcal{H} \rightarrow L^{2}\left(\Omega_{j}\right)$ defined by

$$
A_{j} \mathbf{f}(x):=\left.C \mathbf{f}(x)\right|_{\Omega_{j}}, \quad j=1, \ldots, M
$$

we see that functionals $F_{1}(\mathbf{f})$ and $L(y, \mathbf{f})$ can be rewritten in the form

$$
\begin{gather*}
F_{1}(\mathbf{f})=(\tilde{Q} \mathbf{f}, \mathbf{f})_{\mathcal{H}},  \tag{14}\\
L(y, \mathbf{f})=-2 \Re\left(\mathbf{f}, \sum_{j=1}^{M} A_{j}^{*} C_{j}^{*} D_{j} y_{j}+Q \mathbf{f}_{0}\right)_{\mathcal{H}},
\end{gather*}
$$

where $\tilde{Q}$ and $Q: \mathcal{H} \rightarrow \mathcal{H}$ are Hermitian positive definite bounded operators defined by

$$
\begin{equation*}
\tilde{Q}=\sum_{j=1}^{M} A_{j}^{*} C_{j}^{*} D_{j} C_{j} A_{j}+Q \tag{15}
\end{equation*}
$$

and

$$
Q \mathbf{f}:=\left(\chi_{D_{0}}(\cdot) Q_{1} f, Q_{2} g\right) \quad \forall \mathbf{f}=(f, g) \in \mathcal{H}
$$

respectively, $A_{j}^{*}: L^{2}\left(\Omega_{j}\right) \rightarrow \mathcal{H}$ are the operators adjoint of $A_{j}$ defined by

$$
A_{j}^{*} v=\left(\chi_{D_{0}}(\cdot) \int_{\Omega_{j}} v(x) \overline{\Phi_{k}(x, \cdot)} \mathrm{d} x,\left.\int_{\Omega_{j}} v(x) \overline{\Phi_{k}(x, \cdot)} \mathrm{d} x\right|_{\Gamma}\right) \quad \forall v \in L^{2}\left(\Omega_{j}\right) .
$$

From here, it follows that $F_{1}(\mathbf{f})$ is a quadratic form which corresponds to a semi-linear continuous Hermitian form

$$
\pi(\mathbf{f}, \mathbf{g}):=(\tilde{Q} \mathbf{f}, \mathbf{g})_{\mathcal{H}}
$$

and $L(\mathbf{f})$ a linear continuous functional defined on $\mathcal{H}$. Moreover, since $F_{1}(\mathbf{f})$ is also a strictly convex functional in the space $\mathcal{H}$ satisfying the condition

$$
F_{1}(\mathbf{f}) \geq c\|\mathbf{f}\|_{\mathcal{H}}^{2} \quad \forall \mathbf{f} \in \mathcal{H}, \quad c=\text { const },
$$

we obtain, using Remark 1.1 to Theorem 1.1 from [8], that there exists a unique element $\hat{\mathbf{f}} \in \mathcal{H}$ such that

$$
\inf _{\mathbf{f} \in \mathcal{H}} F(y, \mathbf{f})=F(y, \hat{\mathbf{f}}) .
$$

Hence, for $\tau \in \mathbb{R}$, the following relation is valid

$$
\left.\frac{\mathrm{d}}{\mathrm{~d} \tau} F(y, \hat{\mathbf{f}}+\tau \mathbf{w})\right|_{\tau=0} \equiv 0 \quad \forall \mathbf{w}=\left(w_{1},, w_{2}\right) \in \mathcal{H}
$$

Next, observing that

$$
\begin{align*}
\frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} \tau} F\left(y, \hat{\mathbf{f}}+\left.\tau \mathbf{w}\right|_{\tau=0}=\right. & -\sum_{j=1}^{M} \Re \int_{\Omega_{j}} D_{j}(x)\left(y_{j}(x)-C_{j} \hat{\varphi}(x)\right) \overline{C_{j} \tilde{\varphi}(x)} \mathrm{d} x \\
& +\Re \int_{D_{0}} Q_{1}\left(\hat{f}-f_{0}\right)(x) \overline{w_{1}(x)} \mathrm{d} x \\
& +\Re \int_{\Gamma} Q_{2}\left(\hat{g}-g_{0}\right) \bar{w}_{2} \mathrm{~d} \Gamma \tag{16}
\end{align*}
$$

where $\hat{\varphi}$ and $\tilde{\varphi}$ uniquely solve problem (1)-(3) at $\mathbf{f}=\hat{\mathbf{f}}$ and $\mathbf{f}=\mathbf{w}$, respectively, and introducing function $\hat{p} \in H_{\mathrm{loc}}^{1}\left(D^{\prime}, \Delta\right)$ as the unique solution to the problem

$$
\begin{align*}
-\left(\Delta+\bar{k}^{2}\right) \hat{p}(x) & =\sum_{j=1}^{M} \chi_{\Omega_{j}}(x) C_{j}^{*} D_{j}\left[y_{j}-C_{j} \hat{\varphi}\right](x) \quad \text { in } D^{\prime}  \tag{17}\\
\frac{\partial \hat{p}(\cdot ; u)}{\partial \nu} & =0 \quad \text { on } \Gamma  \tag{18}\\
\frac{\partial \hat{p}}{\partial r}+i \bar{k} \hat{p} & =o\left(1 / r^{(n-1) / 2}\right), \quad r=|x|, \quad r \rightarrow \infty \tag{19}
\end{align*}
$$

we obtain from (16),

$$
\begin{align*}
& \frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} \tau} F\left(y, \hat{\mathbf{f}}+\left.\tau \mathbf{w}\right|_{\tau=0}\right. \\
& =\left\{\left(-\int_{D_{0}} \hat{p}(x) \overline{w_{1}(x)} \mathrm{d} x-\int_{\Gamma} \hat{p} \overline{w_{2}} \mathrm{~d} \Gamma\right.\right. \\
& \left.\quad+\int_{D_{0}} Q_{1}\left(\hat{f}-f_{0}\right)(x) \overline{w_{1}(x)} \mathrm{d} x+\int_{\Gamma} Q_{2}\left(\hat{g}-g_{0}\right) \bar{w}_{2} \mathrm{~d} \Gamma\right) \equiv 0 \quad \forall w \in \mathcal{H} \tag{20}
\end{align*}
$$

Indeed, choosing $R$ large enough so that $\bar{D}, \bar{D}_{0}, \Omega_{j} \subset \Omega_{R}, j=1, \ldots, M$, and applying to $\hat{p}(x)$ and $\tilde{\varphi}(x)$ in the domain $\Omega_{R} \backslash \bar{D}$ the second Green's formula, transforms the first term in the right-hand side of (16). We have

$$
\begin{align*}
& -\sum_{j=1}^{M} \Re \int_{\Omega_{j}} D_{j}(x)\left(y_{j}(x)-C_{j} \hat{\varphi}(x)\right) \overline{C_{j} \tilde{\varphi}(x)} \mathrm{d} x \\
& \quad=-\Re \int_{\Omega_{R} \backslash \bar{D}} \sum_{j=1}^{M} \chi_{\Omega_{j}}(x) C_{j}^{*} D_{j}\left[y_{j}-C_{j} \hat{\varphi}\right](x) \overline{\tilde{\varphi}(x)} \mathrm{d} x \\
& \quad=\Re \int_{\Omega_{R} \backslash \bar{D}}\left(\Delta+\bar{k}^{2}\right) \hat{p}(x) \overline{\tilde{\varphi}(x)} \mathrm{d} x \\
& \quad=\Re\left(\int_{\Omega_{R} \backslash \bar{D}} \hat{p}(x) \overline{\left(\Delta+k^{2}\right) \tilde{\varphi}(x)} \mathrm{d} x-\int_{\Gamma} \hat{p} \frac{\partial \overline{\tilde{\varphi}}}{\partial v} \mathrm{~d} \Gamma+\int_{\Gamma_{R}} \hat{p} \frac{\partial \overline{\tilde{\varphi}}}{\partial v} \mathrm{~d} \Gamma_{R}-\int_{\Gamma_{R}} \frac{\partial \hat{p}}{\partial v} \overline{\tilde{\varphi}} \mathrm{~d} \Gamma_{R}\right) \\
& \quad=\Re\left(-\int_{D_{0}} \hat{p}(x) \overline{w_{1}(x)} \mathrm{d} x-\int_{\Gamma} \hat{p} \bar{w}_{2} \mathrm{~d} \Gamma+\Sigma_{R}(\hat{p}, \tilde{\varphi})\right) \tag{21}
\end{align*}
$$

where by $\Sigma_{R}(\hat{p}, \tilde{\varphi})$ we denote

$$
\Sigma_{R}(\hat{p}, \tilde{\varphi}):=\int_{\Gamma_{R}}\left(\hat{p} \frac{\overline{\partial \tilde{\varphi}}}{\partial v}-\frac{\partial \hat{p}}{\partial \nu} \overline{\tilde{\varphi}}\right) \mathrm{d} \Gamma_{R}
$$

with $\Gamma_{R}=\partial \Omega_{R}, v$ denotes the outward unit normal to the sphere $\Gamma_{R}$. Since $\hat{p}$ and $\tilde{\varphi}$ satisfy, respectively, the Sommerfeld radiation conditions (19) and (3), $\hat{p}(x)=O(1 / R)$ and $\tilde{\varphi}(x)=$
$O(1 / R), R=|x| \rightarrow \infty$ (see [9]), and we obtain an estimate for $\Sigma_{R}(\hat{p}, \tilde{\varphi})$,

$$
\begin{aligned}
\Sigma_{R}(\hat{p}, \tilde{\varphi}) & :=\int_{\Gamma_{R}}\left(\hat{p} \frac{\overline{\partial \tilde{\varphi}}}{\partial v}-i k \tilde{\varphi}\right) \mathrm{d} \Gamma_{R}-\int_{\Gamma_{R}}\left(\frac{\partial \hat{p}}{\partial v}+i \bar{k} \hat{p}\right) \overline{\tilde{\varphi}} \mathrm{d} \Gamma_{R} \\
& =\int_{\Gamma_{R}} O(1 / R) o(1 / R) \mathrm{d} \Gamma_{R}-\int_{\Gamma_{R}} o(1 / R) O(1 / R) \mathrm{d} \Gamma_{R}=o(1) \quad \text { as } R \rightarrow \infty .
\end{aligned}
$$

From here, passing to the limit as $R \rightarrow \infty$ in (21), we obtain (20).
Putting in (20) $w_{1}(x)=Q_{1}\left(\hat{f}-f_{0}\right)(x)-\chi_{D_{0}}(x) \hat{p}(x)$ and $w_{2}=Q_{2}\left(\hat{g}-g_{0}\right)-\left.\hat{p}\right|_{\Gamma}$, we obtain

$$
\begin{equation*}
\int_{D_{0}}\left|Q_{1}\left(\hat{f}-f_{0}\right)(x)-\hat{p}(x)\right|^{2} \mathrm{~d} x+\int_{\Gamma}\left|Q_{2}\left(\hat{g}-g_{0}\right)-\hat{p}\right|^{2} \mathrm{~d} \Gamma=0 \tag{22}
\end{equation*}
$$

Equations (17)-(19) and (22) imply representation (5).
The above analysis and the fact that functional $F(y, \mathbf{f})$ has one minimum point $\hat{\mathbf{f}}$ lead to the conclusion that problem (6)-(10) is uniquely solvable.

Let us prove (12). Let $\vartheta(\tau):=F(y, \hat{\mathbf{f}}+\tau(\mathbf{f}-\hat{\mathbf{f}}))$. By expanding the function $\vartheta(\tau)$ by Taylor's formula in a neighbourhood of zero, we find

$$
\begin{equation*}
\vartheta(\tau)=\vartheta(0)+\left.\frac{\mathrm{d} \vartheta}{\mathrm{~d} \tau}\right|_{\tau=0} \tau+\left.\frac{1}{2} \frac{\mathrm{~d}^{2} \vartheta}{\mathrm{~d} \tau^{2}}\right|_{\tau=0} \tau^{2} . \tag{23}
\end{equation*}
$$

Since $\hat{\mathbf{f}} \in \operatorname{Argmin} F(y, \mathbf{f})$ then $\left.(\mathrm{d} \vartheta / \mathrm{d} \tau)\right|_{\tau=0}=0$. Setting $\tau=1$ in (23), we obtain that

$$
\begin{equation*}
\vartheta(1)=F(y, \mathbf{f})=F(y, \hat{\mathbf{f}})+\left.\frac{1}{2} \frac{\mathrm{~d}^{2} \vartheta}{\mathrm{~d} \tau^{2}}\right|_{\tau=0} . \tag{24}
\end{equation*}
$$

Representation (12) follows from (24) if we observe that $\left.\left(\mathrm{d}^{2} \vartheta / \mathrm{d} \tau^{2}\right)\right|_{\tau=0}=F_{1}(\mathbf{f}-\hat{\mathbf{f}})$. This completes the proof of the lemma.

From this lemma, we conclude that the set $G_{y}$ has the form

$$
\begin{equation*}
G_{y}=\left\{\mathbf{f}: F_{1}(\mathbf{f}-\hat{\mathbf{f}}) \leq \beta_{M}^{2}-F(y, \hat{\mathbf{f}})\right\} . \tag{25}
\end{equation*}
$$

Using this fact, we establish in the proof of Theorem 3.1 that $\mathbf{f}_{g}=\hat{\mathbf{f}}$.
Theorem 3.1: The guaranteed a posteriori estimate $\hat{\mathbf{f}}_{g}$ of unknown vector-function $\mathbf{f}$ coincides with vector-function $\hat{\mathbf{f}}$, i.e.

$$
\hat{f}_{g}(x)=\hat{f}(x) \text { a.e. in } D_{0}, \quad \hat{g}_{g}=\hat{g} \text { a.e. on } \Gamma,
$$

and the guaranteed a posteriori estimate $\hat{\varphi}_{g}$ of unknown solution $\varphi$ coincides with function $\hat{\varphi}$, i.e.

$$
\hat{\varphi}_{g}(x)=\hat{\varphi}(x) \quad \text { a.e. in } D^{\prime}
$$

where $\hat{f}, \hat{g}$ and $\hat{\varphi}$ are determined from (5) to (11).

The estimation error $\delta_{a}$ is determined by

$$
\begin{equation*}
\delta_{a}=\left(\beta_{M}^{2}-F\left(y, \hat{\mathbf{f}}_{g}\right)\right)^{1 / 2} \sup _{\|\mathbf{a}\|_{\mathcal{H}}=1}\left(\tilde{Q}^{-1} \mathbf{a}, \mathbf{a}\right)_{\mathcal{H}}^{1 / 2} \tag{26}
\end{equation*}
$$

where $\tilde{Q}^{-1}$ is the inverse of a Hermitian positive definite bounded operator $\tilde{Q}: \mathcal{H} \rightarrow \mathcal{H}$ defined by (15).

Proof: First, notice that $\tilde{Q}^{-1}$ exists, and it is a Hermitian positive definite bounded operator since it is the inverse of $\tilde{Q}$. It is evident from the definition of element $\hat{\mathbf{f}} \in \mathcal{H}$ that $\hat{\mathbf{f}} \in G_{y}$. From (14) and (25), it follows that the set $G_{y}$ can be represented as

$$
G_{y}=\left\{\mathbf{f} \in \mathcal{H}:(\tilde{Q}(\mathbf{f}-\hat{\mathbf{f}}), \mathbf{f}-\hat{\mathbf{f}})_{\mathcal{H}} \leq \gamma\right\}
$$

where $\gamma:=\beta_{M}^{2}-F(y, \hat{\mathbf{f}})>0$. Introduce also the set $\tilde{G}_{y}=\left\{\tilde{\mathbf{f}} \in \mathcal{H}:(\tilde{Q} \tilde{\mathbf{f}}, \tilde{\mathbf{f}})_{\mathcal{H}} \leq \gamma\right\}$. It is easy to see that Cauchy-Schwarz inequality implies

$$
\left\|\mathbf{f}_{1}-\mathbf{f}_{2}\right\|_{\mathcal{H}}=\sup _{\|\mathbf{a}\|_{\mathcal{H}} \leq 1}\left|\Re\left(\mathbf{a}, \mathbf{f}_{1}-\mathbf{f}_{2}\right)_{\mathcal{H}}\right|
$$

Then for all $\mathbf{f}_{1} \in G_{y}$,

$$
\begin{aligned}
\sup _{\mathbf{f}_{2} \in \mathbf{G}_{\mathbf{y}}}\left\|\mathbf{f}_{1}-\mathbf{f}_{2}\right\|_{\mathcal{H}} & =\sup _{\|\mathbf{a}\|_{\mathcal{H}} \leq 1} \sup _{\mathbf{f}_{2} \in \mathbf{G}_{\mathbf{y}}}\left|\Re\left(\mathbf{a}, \mathbf{f}_{1}\right)_{\mathcal{H}}-\Re\left(\mathbf{a}, \mathbf{f}_{2}\right)_{\mathcal{H}}\right| \\
& =\sup _{\|\mathbf{a}\|_{\mathcal{H}} \leq 1} \sup _{\tilde{\mathbf{f}} \in \tilde{G}_{y}}\left|\Re\left(\mathbf{a}, \mathbf{f}_{1}-\hat{\mathbf{f}}\right)_{\mathcal{H}}-\Re(\mathbf{a}, \tilde{\mathbf{f}})_{\mathcal{H}}\right| \\
& =\sup _{\|\mathbf{a}\|_{\mathcal{H}} \leq 1}\left(\left|\Re\left(\mathbf{a}, \mathbf{f}_{1}-\hat{\mathbf{f}}\right)_{\mathcal{H}}\right|+\sup _{\tilde{\mathbf{f}} \in \tilde{G}_{y}}\left|\Re(\mathbf{a}, \tilde{\mathbf{f}})_{\mathcal{H}}\right|\right) .
\end{aligned}
$$

From here, taking into account the relationship ${ }^{2}$

$$
\sup _{\tilde{\mathbf{f}} \in \tilde{G}_{y}}\left|\Re(\mathbf{a}, \tilde{\mathbf{f}})_{\mathcal{H}}\right|=\gamma^{1 / 2}\left(\tilde{Q}^{-1} \mathbf{a}, \mathbf{a}\right)_{\mathcal{H}}^{1 / 2}
$$

we obtain that for all $\mathbf{f}_{1} \in G_{y}$,

$$
\begin{aligned}
\inf _{\mathbf{f}_{1} \in \mathbf{G}_{\mathbf{y}}} \sup _{\mathbf{f}_{2} \in \mathbf{G}_{\mathbf{y}}}\left\|\mathbf{f}_{1}-\mathbf{f}_{2}\right\|_{\mathcal{H}} & \geq \sup _{\|\mathbf{a}\|_{\mathcal{H}} \leq 1} \inf _{\mathbf{f}_{1} \in \mathbf{G}_{\mathbf{y}}}\left(\left|\Re\left(\mathbf{a}, \mathbf{f}_{1}-\hat{\mathbf{f}}\right)_{\mathcal{H}}\right|+\sup _{\tilde{\mathbf{f}} \in \tilde{G}_{y}}\left|\Re(\mathbf{R}, \tilde{\mathbf{f}})_{\mathcal{H}}\right|\right) \\
& =\sup _{\|\mathbf{a}\|_{\mathcal{H}} \leq 1} \sup _{\tilde{\mathbf{f}}_{\boldsymbol{f}} \in \tilde{G}_{y}}|\mathfrak{R}(\mathbf{a}, \tilde{\mathbf{f}}) \mathcal{H}|=\gamma^{1 / 2} \sup _{\|\mathbf{a}\|_{\mathcal{H}}=1}\left(\tilde{Q}^{-1} \mathbf{a}, \mathbf{a}\right)_{\mathcal{H}}^{1 / 2}
\end{aligned}
$$

where equality is attained at $\mathbf{f}_{1}=\hat{\mathbf{f}}$. Whence, $\hat{\mathbf{f}}_{g}=\hat{\mathbf{f}}, \hat{\varphi}_{g}=\hat{\varphi}$, and $\delta_{a}$ is defined by (26). This completes the proof.

As a corollary, we obtain the following result.

Theorem 3.2: Let $\Omega_{R}:=\left\{x \in \mathbb{R}^{n}:|x|<R\right\}$ and $R_{0}$ be chosen so that $\bar{D} \subset \Omega_{R_{0}}$. Then, for any $R \geq R_{0}$, there exists a positive constant $\alpha_{R}>0$, dependent only on $R$, such that the following inequality holds:

$$
\begin{equation*}
\left\|\varphi-\hat{\varphi}_{g}\right\|_{H^{1}\left(\Omega_{R} \backslash \bar{D}, \Delta\right)} \leq \alpha_{R} \delta_{a}, \tag{27}
\end{equation*}
$$

where $\hat{\varphi}_{g}$ is the guaranteed a posteriori estimate of unknown solution $\varphi, \delta_{a}$ is determined by formula (26), and $\|\psi\|_{H^{1}\left(\Omega_{R} \backslash \bar{D}, \Delta\right)}=\|\psi\|_{H^{1}\left(\Omega_{R} \backslash \bar{D}\right)}+\|\Delta \psi\|_{L^{2}\left(\Omega_{R} \backslash \bar{D}\right)} \forall \psi \in H_{\mathrm{loc}}^{1}\left(D^{\prime}, \Delta\right)$.

Proof: Since $\hat{\varphi}_{g}(x)$ solves problem (1)-(3) at $\mathbf{f}=\hat{\mathbf{f}}_{g}$, function $\varphi_{1}(x):=\varphi(x)-\hat{\varphi}_{g}(x)$ solves the following problem:

$$
\begin{aligned}
-\left(\Delta+k^{2}\right) \varphi_{1} & =f_{1} \quad \text { in } D^{\prime} \\
\frac{\partial \varphi_{1}}{\partial v} & =g_{1} \quad \text { on } \Gamma \\
\frac{\partial \varphi_{1}}{\partial r}-i k \varphi_{1} & =o\left(1 / r^{(n-1) / 2}\right), \quad r=|x|, \quad r \rightarrow \infty
\end{aligned}
$$

where $f_{1}=f-\hat{f}_{g}, g_{1}=g-\hat{g}_{g}$. From a priori estimates for exterior Neumann problem for the Helmholtz equation (see $[6,10]$ ), it follows that

$$
\left\|\varphi_{1}\right\|_{H^{1}\left(\Omega_{R} \backslash \bar{D}, \Delta\right)} \leq \alpha_{R}\left\|\mathbf{f}_{1}\right\|_{\mathcal{H}},
$$

where $\alpha_{R}$ a positive constant dependent only on $R, \mathbf{f}_{1}=\left(f_{1}, g_{1}\right)$, whence it follows estimate (27).

## 4. Conclusion

Following our main objective to establish a technique for obtaining guaranteed a posteriori estimates in acoustic and electromagnetic inverse problems, we have constructed the tools for efficient estimation of the right-hand sides entering Neumann problems for the Helmholtz equation that model the wave fields in acoustic scattering on rigid bodies or electromagnetic scattering on perfectly conducting cylindrical bodies.

We have proposed the relevant mathematically correct definition of guaranteed a posteriori estimate, and the description of measurement errors.

A uniquely solvable linear system of Helmholtz equations has been obtained that generates guaranteed a posteriori estimates of the Neumann data.

The developed approach continue our studies aimed at elaborating mathematically justified solution techniques for various forward and inverse problems with uncertainties arising in electromagnetics and acoustics.

## Notes

1. That is, for fixed $x \in D^{\prime}$ function, $\Phi_{k}(x, y)$ solves the Neumann boundary value problem:

$$
\begin{aligned}
-\left(\Delta_{y}+k^{2}\right) \Phi_{k}(x, y) & =\delta(x-y), \quad y \in D^{\prime}, \\
\frac{\partial \Phi_{k}(x, y)}{\partial v_{x}} & =0, \quad y \in \Gamma, \\
\frac{\partial \Phi_{k}(x, y)}{\partial r}-i k \Phi_{k}(x, y) & =o\left(1 / r^{(n-1) / 2}\right), \quad r=|y|, \quad r \rightarrow \infty
\end{aligned}
$$

where $\Delta_{y}$ denotes the Laplacian with respect to the $y$ variables, and $\delta$ is the Dirac delta function concentrated at $x$.
2. In fact, by virtue of generalized Cauchy-Schwarz inequality [11, p.186],

$$
\sup _{\tilde{\mathbf{f}} \in \tilde{G}_{y}}\left|\Re(\mathbf{R}, \tilde{\mathbf{f}})_{\mathcal{H}}\right| \leq \sup _{\tilde{\mathbf{f}} \in \tilde{\mathbb{G}}_{y}}\left|(\mathbf{a}, \tilde{\mathbf{f}})_{\mathcal{H}}\right| \leq \sup _{\tilde{\mathbf{f}} \in \tilde{\mathbb{G}}_{y}}\left(\tilde{Q}^{-1} \mathbf{a}, \mathbf{a}\right)_{\mathcal{H}}^{1 / 2}(\tilde{Q} \tilde{\mathbf{f}}, \tilde{\mathbf{f}})_{\mathcal{H}}^{1 / 2}=\gamma^{1 / 2}\left(\tilde{Q}^{-1} \mathbf{a}, \mathbf{a}\right)_{\mathcal{H}}^{1 / 2},
$$

and this inequality is transformed into an equality on the element $\tilde{\mathbf{f}}=\gamma^{1 / 2}\left(\tilde{Q}^{-1} \mathbf{a}, \mathbf{a}\right)_{\mathcal{H}}^{-1 / 2} \tilde{Q}^{-1} \mathbf{a}$.

## Disclosure statement

No potential conflict of interest was reported by the authors.

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