

Physical and Geometric Effects on the Classical Geodetic Observations in Small-Scale Control Networks

Mohammad Bagherbandi¹; Masoud Shirazian²; Jonas Ågren³; and Milan Horemuz⁴

Abstract: In classical two-dimensional (2D) geodetic networks, reducing slope distances to horizontal ones is an important task for engineers. These horizontal distances along with horizontal directions are used in 2D geodetic adjustment. The common practice for this reduction is the use of vertical angles to reduce distances using trigonometric rules. However, one faces systematic effects when using vertical angles. These effects are mainly due to refraction, deflection of the vertical (DOV), and the geometric effect of the reference surface (sphere or ellipsoid). To mitigate refraction and DOV effects, one can choose to observe the vertical angles reciprocally if the baseline points' elevation difference is small. This paper quantifies these effects and proposes a proper solution to eliminate the effects in small-scale geodetic networks (where the longest distances are less than 5 km). The goal is to calculate slope distances into horizontal ones appropriately. For this purpose, we used the SWEN17_RH2000 quasigeoid model (in Sweden) to study the impact of the DOV applying different baseline lengths, azimuths, and vertical angles. Finally, we propose an approach to study the impact of the geometric effect on vertical angles. We illustrate that the DOV and the geometric effects on vertical angles measured reciprocally are significant if the height difference of the start point and endpoint in the baseline is large. Geometric correction should be considered for the measured vertical angles to calculate horizontal distances correctly if the network points are not on the same elevation, even if the vertical angles are measured reciprocally. **DOI: 10.1061/(ASCE)SU.1943-5428.0000407.** This work is made available under the terms of the Creative Commons Attribution 4.0 International license, <https://creativecommons.org/licenses/by/4.0/>.

Author keywords: Geodetic network; Deflections of the vertical (DOV); Geometric effects; Normal skewness; Refraction; Vertical angle.

Introduction

Establishing precise geodetic control networks (cf. Schaffrin 1985; Grafarend and Sansò 2012) is one of the most important issues in geodesy and land surveying engineering. It might be thought that designating geodetic networks is a settled issue because of existing space-based (e.g., GNSS) and nongeodetic sensors (e.g., strain-gauge, tiltmeters) technologies, but this is not true in reality. In this paper, we will show that challenges remain and that we need new data processing approaches and precise measurement routines to maintain geodetic control networks. For example, we are still using classical geodetic measurements and methods to monitor dams, landslides (especially in rough topography areas), and tunnels where it is not practical or possible to collect global navigation

satellite systems (GNSS) observations. In addition, classical geodetic observations should be corrected due to error sources before control network adjustment.

There are three main challenges that one should consider when establishing geodetic networks are refraction error (cf. Ashkenazi and Howard 1984; Brunner 1984; Dodson and Zaher 1985; Rapp 1993), physical effects (due to the deflection of the vertical; Heiskanen and Moritz 1967, p. 312), and geometric effects. Geometric correction on the reference sphere is called curvature correction due to the sphere curvature and the curvature-skewness problem on the reference ellipsoid (cf. Krakiwsky and Thomson 1974; Rapp 1991). In other words, there are two effects on the reference ellipsoid: (1) geometric effects due to the surface curvature; and (2) the skewness of the normals. It is well known that curvature is constant on a spherical surface; thus, the skewness error is zero.

The atmospheric refraction error is a particular problem in any optical measurement. The air temperature gradient in the direction perpendicular to the line of sight is the main factor to model the refraction effect. To precisely compute the height differences and reduce the slope distances to the horizontal counterparts, the measured vertical angles must be corrected for the refraction effect. A usual way to overcome this error is to simultaneously observe the vertical angles at the two endpoints of the line (cf. Vanicek and Krakiwsky 1986, p. 380). In other words, we assume that the refraction error is unknown and is the same (with opposite signs) from both ends of the line. One can write the vertical angle observation equations for the two reciprocal observations. After subtracting the observation equations, a new observation equation is obtained in which the refraction error is eliminated. The obtained observation equation can then be used in the geodetic control network adjustment step by considering a weight based on the weights of the initial reciprocal measurements (Rapp 1993).

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Note. This manuscript was submitted on January 27, 2022; approved on May 31, 2022; published online on September 26, 2022. Discussion period open until February 26, 2023; separate discussions must be submitted for individual papers. This paper is part of the *Journal of Surveying Engineering*, © ASCE, ISSN 0733-9453.

Definition of a proper coordinate system is one of the most important problems in establishing a classical geodetic network. The common practice is to establish either a local geodetic or astronomic coordinate system (Vanicek and Krakiwsky 1986). Physical and geometric problems occur when the up-component directions (i.e., plumb line or normal direction) are different at the start and endpoints of a baseline (because they are not coplanar assuming actual earth shape and an ellipsoidal model).

Most conventional geodetic measurements refer to the plumb line, that is, the direction of the gravity vector (Featherstone and Rüeger 2000). Therefore, the angle between the gravity vector and the ellipsoidal normal, which is called the deflection of the vertical (DOV) or physical effect, should be considered in the geodetic observation reduction step (see also Heiskanen and Moritz 1967, pp. 174 and 312).

Finally, the geometric effect, that is, curvature correction on the sphere or curvature-skewness problem on the reference ellipsoid, should also be treated in the data reduction step. Therefore, errors due to the refraction, DOV, and curvature-skewness problems influence the vertical angles, and one should seriously consider them, especially if the height differences between baseline endpoints are large. For example, a practical solution to reduce refraction error is collecting geodetic measurements using the reciprocal technique either simultaneously or at the same time of day such that refraction effects can be assumed to be the same (cf. Rapp 1993). Nevertheless, turbulent flow in the atmosphere may occur on all spatial and very short temporal scales. Even simultaneous reciprocal observations might not eliminate it. However, collecting reciprocal observations is the only practical solution and is time consuming, especially in rough topography areas (e.g., dam sites), and increases the fieldwork and cost of projects. Cost and time are important factors to establish optimum and precise geodetic networks (Kuang 1996).

As mentioned before, the physical and geometric effects mainly influence the vertical angle that is an important observation to convert slope distances to horizontal ones using the trigonometric method (Schofield and Breach 2007). To the best of the authors' knowledge, these problems are currently usually not clearly mentioned in guidelines and technical specifications. Hence, one of our motivations is to quantify the previously mentioned effects. Second, we highlight the methods one can avoid using vertical angles to determine the horizontal distances, that is, the network-aided method introduced by Shirazian et al. (2021). Therefore, the geometric, physical, and refraction effects will no longer be a matter of concern. This paper aimed to study the reduction of classical geodetic observations due to the geometric and physical effects that directly influence measured vertical angles and indirectly influence distance measurements. In other words, our goal is to reduce slope distances into horizontal ones appropriately in geodetic control networks.

In brief, this study consisted of the following steps. First, the impact of DOV on slope distance reduction is presented in the "Method" section. To perform this investigation, we used the Swedish precise SWEN17 quasigeoid model (Ågren et al. 2018) as a basis to study the impact of the DOV on horizontal distances in geodetic control networks. We also compared the obtained DOV with the one obtained by the EGM2008 model (Pavlis et al. 2012) and astronomical observations in Sweden. Finally, the estimated DOV from the SWEN17 model was used to study the impact of DOV on slope distance reduction. Second, we present the geometric effect (curvature-skewness error) considering both ellipsoidal and spherical Earth models. The impact of the geometric effect has been presented only for horizontal directions in the classical geodesy literature, for example, in Krakiwsky and Thomson (1974),

Bomford (1971), and Rapp (1991). Here, we propose a new technique to quantify the geometric effect on vertical angles by performing a simulation that can be relevant and useful for the writing of surveying guidelines. We simulated the impact of the curvature-skewness error on the slope distance reduction using different baseline lengths and height differences. A comprehensive discussion and presentation of the DOV and curvature-skewness effect are given in the "Results" section.

Method

Deflection of the Vertical Effect on Geodetic Observations

The deflection of the vertical at the geoid surface is usually defined as a spatial angle (ε) between the normals to the geoid and ellipsoid surface (Fig. 1). The DOV thus also shows the slope of the geoid with respect to the Earth's reference ellipsoid (Vanicek and Krakiwsky 1986; Sjöberg and Bagherbandi 2017). The DOV angle ε deviates from ε' elsewhere due to the local features of the Earth's gravity field and the curvature of the plumb line (cf. Vanicek and Krakiwsky 1986, pp. 491–506). In Fig. 1, ε' denotes the DOV at the Earth's surface.

The well-known equations of DOV components in the meridian and prime vertical directions at any point of interest on the geoid surface can be found in Heiskanen and Moritz (1967, p. 312). Corresponding equations can be written for the DOV components at the Earth's surface using the quasigeoid and Molodenskij's definition of the height anomaly (ζ). Molodenskij et al. (1962) discarded the geoid and defined a new surface, the quasigeoid, in which geoidal undulation is replaced by height anomaly. The north-south (ξ') and east-west (η') components of the DOV are obtained by Heiskanen and Moritz (1967, p. 313)

$$\xi' = -\frac{1}{R} \frac{\partial \zeta}{\partial \varphi} - \frac{\Delta g}{\gamma} \frac{1}{R} \frac{\partial H}{\partial \varphi} \quad (1a)$$

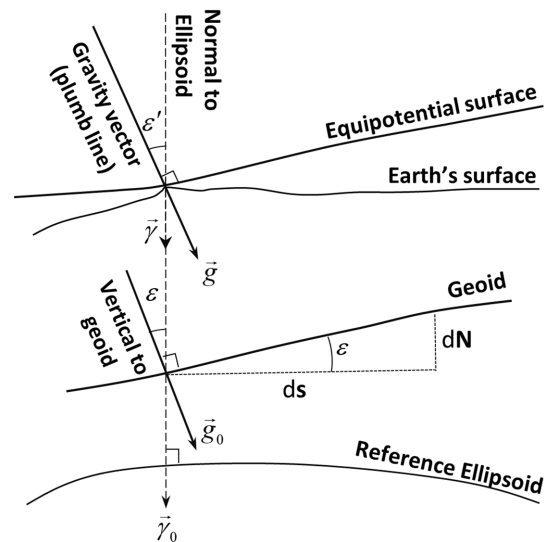


Fig. 1. Basic principle of the deflection of the verticals: \vec{g}_0 : actual gravity on the geoid, $\vec{\gamma}_0$: normal gravity on the reference ellipsoid, \vec{g} : actual gravity vector on the Earth's surface, and $\vec{\gamma}$: normal gravity vector on the Earth's surface; dN and ds are infinitesimal changes of geoid height and distance, respectively.

$$\eta' = -\frac{1}{R \cos \varphi} \frac{\partial \zeta}{\partial \lambda} - \frac{\Delta g}{\gamma} \frac{1}{R \cos \varphi} \frac{\partial H}{\partial \lambda} \quad (1b)$$

where R = mean radius of the Earth (the radius of a sphere whose volume equals that of the ellipsoid according to Moritz 2000); φ and λ = latitude and longitude of the desired point, respectively; γ = normal gravity on the earth's surface; H = topographic height (normal height in this case study because there is quasigeoid model in Sweden); and Δg = gravity anomaly (Heiskanen and Moritz 1967, p. 83). Therefore, Eq. (1) provides us with the fundamental links between the Stokes and Molodenskij approaches, which involve the topographical and curvature of plumb line effects (Heiskanen and Moritz 1967, p. 312; Vanicek and Krakiwsky 1986, p. 533). The formulae of determination of the DOV components using an Earth gravitational model (e.g., the EGM2008 model) are also presented in Supplemental Materials.

The curvature of the plumb line correction should be considered, especially in rough topography areas. This correction was presented by Vanicek and Krakiwsky (1986, p. 506) and Jekeli (1999). They showed that it mainly affects the north–south component, that is, ξ . In other words, the correction depends on the normal gravity field that does not change in the east–west direction. Therefore, the plumb line curvature does not affect the east–west component. See more details in Vanicek and Krakiwsky (1986, p. 506) and Jekeli (1999).

As already mentioned, conventional geodetic observations are affected by the physical effect, that is, by the DOV components. In other words, the gravity field corrections (i.e., corrections for the effects of the DOVs and geoidal undulations) should be considered before performing geodetic control network adjustment if vertical angles (zenith angles) are used to convert the slope distances to horizontal distances (cf. Shirazian et al. 2021).

After calculating ξ' and η' , one can calculate the effect of DOV on zenith angles (δ_{Z_P}) at the Earth's surface (Point P) using the following equation [Heiskanen and Moritz 1967, Eqs. (5)–(16)]:

$$\delta_{Z_P} = \xi'_P \cos(\alpha_{PQ}) + \eta'_P \sin(\alpha_{PQ}) \quad (2)$$

where α_{PQ} = geodetic azimuth from Point P to Point Q . Eq. (2) shows the projection of the DOV onto the plane that goes through Points P and Q and the center of the Earth (for a spherical earth model but not for an ellipsoidal model) with geodetic azimuth α_{PQ} . The slope distance (SD) is converted to the horizontal distance (HD) using the following equation:

$$\text{HD} = \text{SD} \cos(90 - Z) = \text{SD} \sin Z \quad (3a)$$

where Z = observed vertical (zenith) angle. Finally, the effect of δ_{Z_P} on the horizontal distances can be determined by (i.e., considering with and without the DOV effect on the vertical angle)

$$\delta_{\text{HD}} = \text{SD} \sin(Z + \delta_{Z_P}) - \text{SD} \sin(Z) \quad (3b)$$

Geometric Effect on the Slope Distance Reduction

The geometric effect in question causes curvature-skewness and curvature errors on the ellipsoid [Fig. 2(a)] and sphere [Fig. 2(b)], respectively. In order to derive the geometric effect on slope distances, the Earth is considered first as an ellipsoid of revolution and then as a sphere. This is illustrated in Figs. 2(c and d) with more details. For example, the normals at two distinct points on the ellipsoid do not generally lie in the same plane (they are skewed) and never coincide (except if the azimuth is zero or 90°). In other

words, there is more than one definition of horizontal distance between Points P and Q . This means that the horizontal distance from P to Q is not equal to the horizontal distance from Q to P due to the geometric effect (cf. Rollins and Meyer 2019). If we considered P and Q points on different meridians, then the normals pass through n_P and n_Q , respectively (n_P and n_Q are the intersection of normals at P and Q with the Z -axis). Hence, assuming the ellipsoid model [Figs. 2(a and c)], the effect of the curvature of the reference surface is the difference between horizontal distance PQ (computed in the horizontal plane of P using the zenith angle) and the ellipsoidal distance P_1Q_1 (i.e., the geodesic length between P_1 and Q_1). The effect of skewness problem is the difference between ellipsoidal distances P_1Q_1 and $P_1Q'_1$. The skewness is quantified by the angle $Q_1-Q-Q'_1$ (shown by angle θ) and the curvature-skewness by angle β [assuming the spherical model, the angle between the normals is shown by β' ; see Fig. 2(d)]. This geometric property is called the curvature-skewness angle and affects the horizontal and vertical angles and distances indirectly (using the vertical angle to reduce slope to horizontal distances). It is important to emphasize that the angle β shows the nonparallelism of normals at Points P and Q [because the normals at P and Q are not in the same plane; see Fig. 2(a)]. In other words, as can be seen in Figs. 2(a and c), one should not assume any angle between two normals at Points P and Q because they are not at the same normal plane. In practice (for simplicity), one can define an angle β that is the angle between the normals at P and Q projected to the plane of the normal section between P and Q (see also Fig. 3).

In the following, we present the impact of the curvature-skewness problem on horizontal and vertical angles. The effect of the curvature-skewness error on the horizontal angles is explained in Rapp (1991). We call this effect $\delta_{\alpha_{PQ}}$ for the two Points P and Q , and it is

$$\delta_{\alpha_{PQ}} = \frac{\Delta h}{\hat{N}_P} \Psi_P^2 \left(\sin \alpha_{PQ} \cos \alpha_{PQ} - \frac{s_{PQ}}{2\hat{N}_P} \sin \alpha_{PQ} \tan \varphi_P \right) \quad (4a)$$

where Δh = ellipsoidal height difference between the two points; α_{PQ} = azimuth from P_1 to Q_1 ; s_{PQ} is the length of the geodesic between P_1 to Q_1 ; φ_P is the latitude of P ; and \hat{N}_P (radius of curvature in the prime vertical at P) and Ψ_P^2 are as follows:

$$\hat{N}_P = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi_P}} \quad (4b)$$

$$\Psi_P^2 = (e')^2 \cos^2 \varphi_P \quad (4c)$$

where $e^2 = (a^2 - b^2)/a^2$; $(e')^2 = (a^2 - b^2)/b^2$; and a and b = semimajor and semiminor axes of the ellipsoid. The reader should be aware that Eq. (4a) is not useable if P is not at one of the poles: $\alpha_{PQ} = 0$, so $\sin \alpha_{PQ} = 0$ and $\delta_{\alpha_{PQ}} = 0$. For $s_{PQ} = 2,000$ m, $\varphi_P = \alpha_{PQ} = 45^\circ$, and $h = 100$ m on the GRS80 ellipsoid (Moritz 2000), $\delta_{\alpha_{PQ}}$ is about 0.005 (in arc seconds), which is negligible in small-scale networks. As mentioned before, the curvature-skewness problem has been presented only for horizontal angles in the classical geodesy literature (Krakiwsky and Thomson 1974; Rapp 1991). Hence, we present a method to quantify this problem for vertical angles, which is important for converting from slope to horizontal distances. We will show that the effect of the curvature-skewness error on vertical angles is significant and must be accounted for when dealing with precise geodetic networks. A schematic illustration of the geometric effect (corresponding to the curvature-skewness problem, assuming the ellipsoidal model) on vertical angles is shown in Fig. 3 (considering the horizontal plane passing Point P). Here we assume that the angle β in Fig. 2(c) is the angle

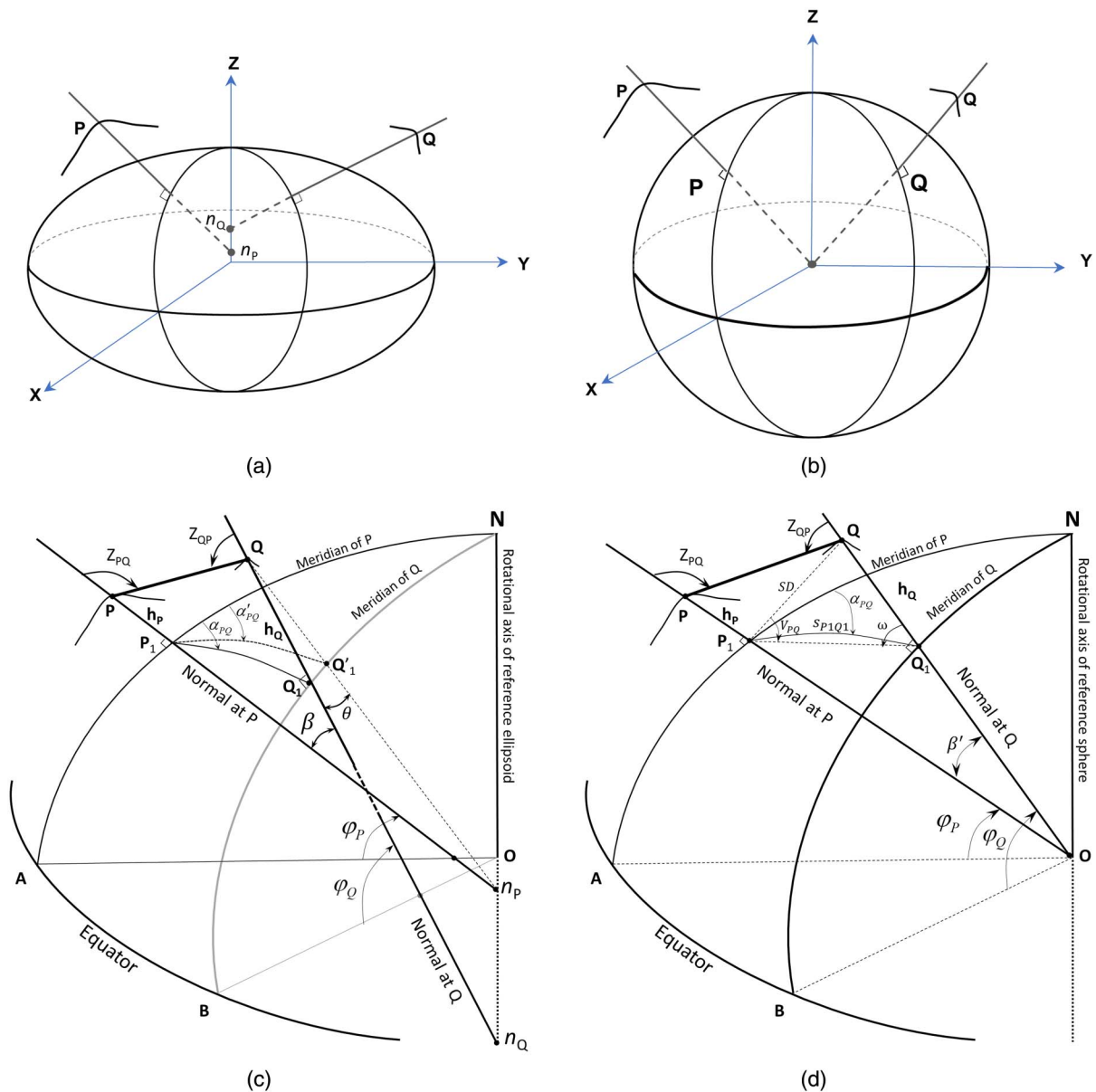


Fig. 2. (a) Curvature-skewness effect (ellipsoidal model); (b) curvature effect (spherical model); (c) sectional view of geometric effect at Points P and Q above an ellipsoid of revolution; and (d) sectional view of geometric effect at Points P and Q above a sphere.

between the normals at P and Q projected to the plane of the normal section between P and Q .

As in Fig. 3, the slope angle at Point P is

$$V_{PQ} = V = 90 - Z \quad (5a)$$

and this angle at Point Q is

$$V_{QP} = -V - \beta \quad (5b)$$

Thus, a unidirectional observing vertical angle will have an error of β due to the curvature-skewness effect. If one measures the vertical angle reciprocally and takes the average of the measurements at the two endpoints of the line, then average reads

$$\bar{V} = \frac{1}{2}(V_{PQ} - V_{QP}) = V + \frac{\beta}{2} \quad (5c)$$

This means that even in the reciprocal measurement of the vertical angles, the effect of the curvature-skewness problem does not disappear, and half of that remains in the measurement. This imposes an error in the slope distance reduction, which is

$$\delta_{HD}^{C-S} = SD \cos(V) - SD \cos(\bar{V}) = SD \left[\cos(V) - \cos\left(V + \frac{\beta}{2}\right) \right] \quad (6)$$

A method is proposed in this paper to compute the curvature-skewness (C-S) error in Eq. (6) accurately. This method consists of the following steps:

In the first step, we establish a local geodetic coordinate system (east-north-up or *enu* system; see Fig. 4) at Point P and compute the coordinate differences, that is, Δe_{PQ} , Δn_{PQ} , and Δu_{PQ} . Then, the slope angle from Point P to Point Q will be

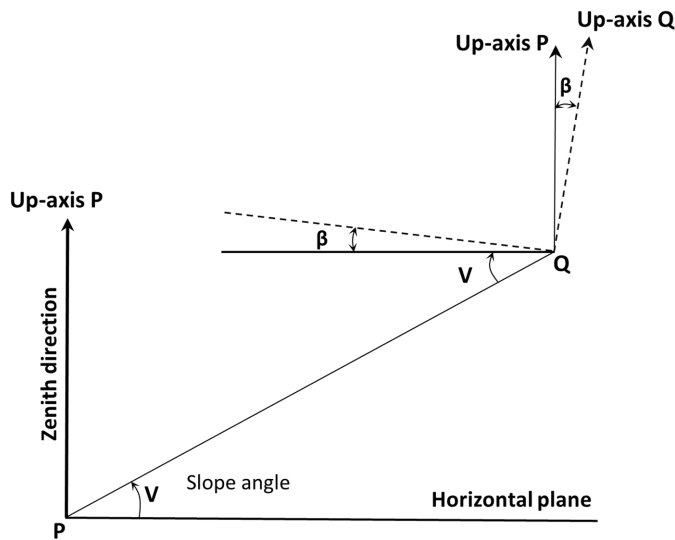


Fig. 3. A schematic illustration of the geometric effect on the vertical angles (V : slope angle, up-axis P : zenith direction at Point P , and up-axis Q : zenith direction at Point Q).

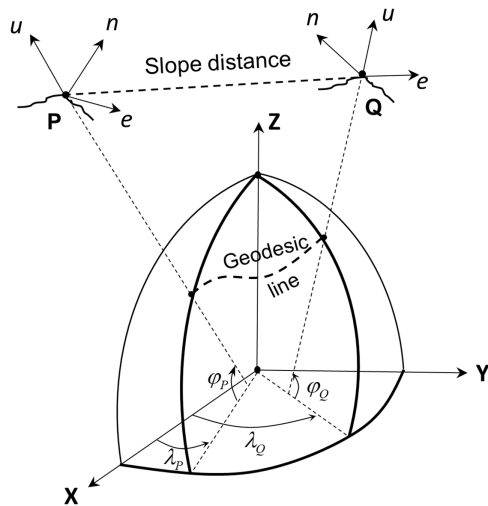


Fig. 4. The *enu* coordinate systems at Points P and Q .

$$V_{PQ} = V = \tan^{-1} \left(\frac{\Delta u_{PQ}}{\sqrt{\Delta e_{PQ}^2 + \Delta n_{PQ}^2}} \right) \quad (7a)$$

To determine the coordinates (at Points P and Q), one can employ the Vincenty algorithm (Vincenty 1975) or the method proposed by Bowring (1983). Using the Vincenty formula and assuming a reference ellipsoid (GRS80), we computed the endpoint coordinates along with the backward azimuth with submillimeter accuracy for our purpose, starting from the geodesic length between two points, forward azimuth, and coordinates of the starting point (direct problem in Geodesy).

In the second step, similarly, we established another *enu* system at Point Q and computed Δe_{QP} , Δn_{QP} , and Δu_{QP} . Then, the slope angle from Point Q to Point P will be

$$V_{QP} = V + \beta = \tan^{-1} \left(\frac{\Delta u_{QP}}{\sqrt{\Delta e_{QP}^2 + \Delta n_{QP}^2}} \right) \quad (7b)$$

Finally, one can compute the curvature-skewness effect on the horizontal distances (δ_{HD}^{C-S}) according to Eqs. (5c) and (6).

Therefore, if we measure the vertical angles reciprocally, we have to add δ_{HD}^{C-S} from Eq. (6) to $SD \cos(V)$ to get the horizontal distance in the horizontal plane of P . In practice, we have to choose a common reference plane (or generally a surface) for all distances in the network.

Assuming a spherical approximation of the ellipsoid, the curvature effect on the slope distance reduction can be obtained by the following steps. First, the average of the latitudes of Points P and Q is considered to calculate the mean radius of curvature, that is, Gaussian curvature (cf. Heiskanen and Moritz 1967)

$$R_m = \frac{b}{1 - e^2 \sin^2 \varphi} \quad (8a)$$

Then, the angle between two normals (at Points P and Q) can be obtained by [Fig. 2(d)]

$$\beta' = \frac{s_{P_1Q_1}}{R_m} \quad (8b)$$

We denote the first point (i.e., P) as projected on the sphere surface ($h_P = 0$); then, it will become P_1 . Thus, the slope distance between Points P_1 and Q is given by

$$SD = \sqrt{R_m^2 + (R_m + h_Q)^2 - 2R_m(R_m + h_Q) \cos \beta'} \quad (8c)$$

The slope angle on the sphere at Point P_1 is obtained by [see the planar triangle P_1QQ_1 in Fig. 2(d)]

$$V_{P_1Q}^{\text{spherical}} = \arcsin \left(\frac{h_Q \sin \omega}{SD} \right) \quad (8d)$$

where $\omega = (\pi + \beta')/2$. Finally, the curvature error on the slope distance reduction is obtained by

$$\delta_{HD}^{\text{Curvature}} = SD \cos \left(V_{P_1Q}^{\text{spherical}} + \frac{\beta'}{2} \right) - SD \cos(V_{P_1Q}^{\text{spherical}}) \quad (8e)$$

Results

This section briefly describes the input data acquisition and the results obtained by applying methods for studying the reduction of classical geodetic observations due to the geometric and physical effects.

Data

The impact of the DOV (physical effect) should be assessed and considered in geodetic control networks. To assess the effect of the DOV on geodetic observations (e.g., vertical angles) in the geodetic network, we need precise data. In this paper, we selected Sweden as a study area because there are high-quality regional gravity data, geoid, and photogrammetric digital elevation models in that region. Our study area lies between latitudes 54.5°N and 69.5°N and longitudes 10.5°E and 24.5°E. According to Ågren et al. (2018), the standard uncertainty of the SWEN17_RH2000 quasi-geoid model has been estimated to be 8–10 mm except for a few

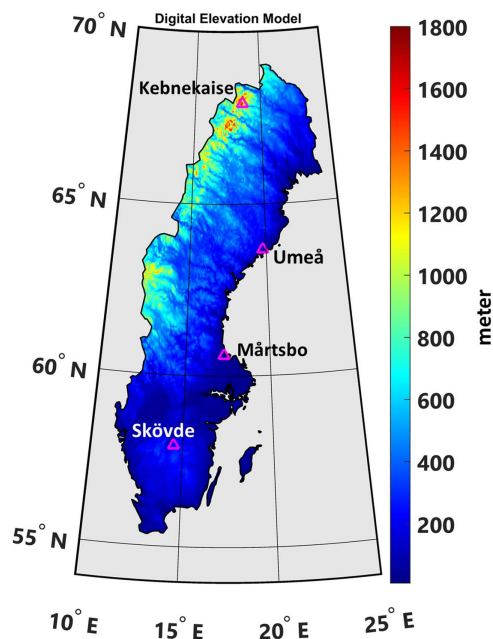


Fig. 5. The Swedish photogrammetric digital elevation model. Triangles show the locations of the selected points for this study, Kebnekaise, Umeå, Mårtsbo, and Skövde, from north to south in Sweden, respectively. Unit: m.

areas where the uncertainty is larger (up to 2–4 cm). Hence, we studied the influence of the DOV using the Swedish national quasigeoid model. Fig. 5 shows the Swedish national elevation model available via the Lantmäteriet (Swedish Mapping, Cadastral and Land Registration Authority) website. This model is stored in a 50-m grid format (Lantmäteriet 2020) and was produced during 2009–2017 using airborne laser scanning. The four triangles in Fig. 5 show the locations of the stations that are used to study

the DOV effect on the slope distance reduction to the horizontal distance. The resolutions of the quasigeoid in the latitude and longitude directions were 0.01° and 0.02° ; thus, we resampled the elevation to the same resolution as the quasigeoid model.

Deflection of Vertical Effect on Horizontal Distances

In this section, first, we present the determined DOV components at the Earth's surface obtained from the SWEN17_RH2000 quasigeoid model and Eqs. (1a) and (1b), and then the DOV impact on the classical geodetic observations will be assessed. The north–south (ξ) and east–west (η) components in Sweden are shown in Fig. 6. We used the same abbreviation as the quasigeoid model (i.e., SWEN17) for the obtained DOV components. Generally, large values for the DOV components can be seen in Lapland (northwest of Sweden), Västerbotten, Ångermanland, Medelpad (northeast of Sweden), and Småland (south of Sweden).

The calculated DOV components can also be compared with one determined using spherical harmonic coefficients, such as the EGM2008 model (Pavlis et al. 2012), at the Earth's surface (see Supplemental Materials). The statistics of the DOVs obtained from the EGM2008 and SWEN17 models in Sweden are presented in Table 1. The EGM2008 and SWEN17 models show that the DOVs are generally similar except for the rough topography areas (especially in the northwest of Sweden) because the EGM2008 model has a lower resolution than SWEN17_RH2000 (see also Fig. S1 in Supplemental Materials). Hence, we studied the impact of the DOV on geodetic observations using only SWEN17 in this paper.

Table 2 shows the magnitude of the north–south (ξ) and east–west (η) components at the selected points in Kebnekaise, Umeå, Mårtsbo, and Skövde. We selected these stations in different latitudes (from the north to the south) to scrutinize the influence of the DOV on geodetic observation in different regions. The Kebnekaise station is located in the northwest part of Sweden that has the highest mountains. The Umeå and Mårtsbo stations are close to the shorelines, with close to minimum height anomaly values

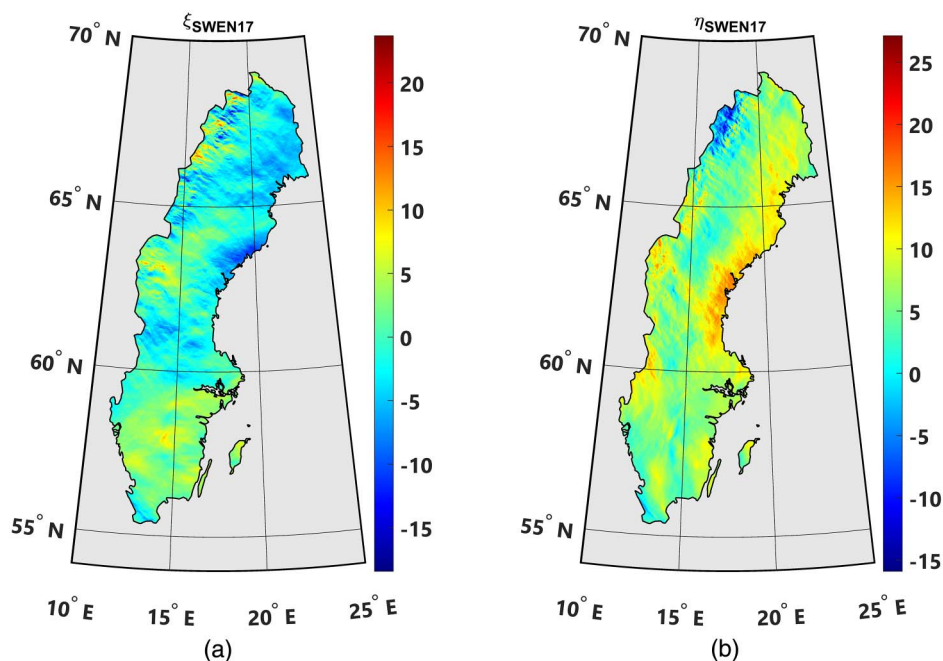


Fig. 6. The deflection of vertical components [(a) ξ ; and (b) η] at the Earth's surface with a grid resolution of $0.01^\circ \times 0.02^\circ$ in latitude and longitude directions in Sweden. Unit: arc second.

Table 1. Statistics of deflection of the vertical using SWEN17 and EGM2008 models and their differences (denoted by Δ) in Sweden. Unit: arc second

DOV component	Max	Mean	Min	STD
ξ_{EGM2008}	18.9	-0.4	-14.7	3.9
η_{EGM2008}	25.8	6.2	-10.9	3.6
ξ_{SWEN17}	23.5	0.4	-18.6	3.8
η_{SWEN17}	27.1	6.2	-15.9	4.1
$\Delta\xi$	12.9	0.0	-13.5	1.1
$\Delta\eta$	12.1	0.0	-11.7	1.1

(on the Swedish mainland). Finally, we selected the Skövde station in the south of Sweden, representing a medium value of height anomaly in the study area. We believe that this combination can be a good sample to examine the effect of the DOV.

To evaluate the obtained DOV components, one can compare them with those obtained from classic astronomical observations or advanced zenith camera systems [e.g., see (Hirt and Seeber 2002)]. At the time of writing the present paper, we had access only to the classic astronomical observations through Ekman and Ågren's (2010) report (see also Wargentín 1759; Cronstrand 1811; Selander 1835). The comparison between the DOVs obtained by the SWEN17 model and astronomical observations can be seen in Table S1 in the Supplemental Materials.

We aim to show the effect of the DOV on the reduction of the slope distances to the horizontal ones and conclude whether its

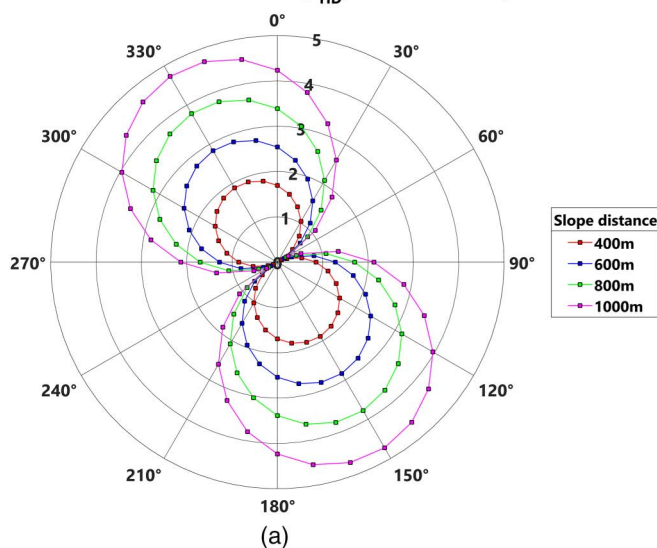
impact is significant (in the case of using vertical angles to convert slope to horizontal distances). Using Eq. (3b), one can calculate the impact of the DOV on the slope distance reduction by assuming a value for the zenith angle. According to the surveying guidelines and literature (e.g., USACE 2002, pp. 2–13; Krakiwsky and Thomson 1974, p. 37), it has been recommended to design the geodetic control points so that the stations' elevations are as similar as possible in order to eliminate or reduce both physical and geometric effects (i.e., the DOV and curvature-skewness problem). Therefore, the zenith angles should usually be as close to 90° as possible and, in practice, vary within $90^\circ \pm 5^\circ$ to fulfil this criterion. However, this assumption causes another problem, atmospheric refraction error, because the observed zenith angles are very sensitive to the atmospheric refraction if the zenith angle varies between 90° and 85° (cf. Vanicek and Krakiwsky 1986, p. 366). Zenith angles close to 90° will, on the other hand, be an ideal case for the DOV and curvature-skewness effects. However, we assumed 85° for the zenith angle in our simulation. The results of the impact of the DOV on horizontal distances can be seen in Figs. 7–10 in the selected stations (i.e., Kebnekaise, Umeå, Mårtsbo, and Skövde). We plotted the absolute values of δ_{HD} considering different baselines (varying between 0.4 and 5 km) and azimuths (varying between 0° and 360° with 10° intervals). As we show in Tables S2–S5 in the Supplemental Materials, the impact of the DOV on horizontal distances will be as large on the opposite side (i.e., 180° azimuth difference). This is the reason for illustrating the absolute values of δ_{HD} in Figs. 7–10. In addition, similar plots are presented in Figs. S2–S5 (see Supplemental Materials) assuming 70° for zenith

Table 2. The magnitude of the deflection of vertical components at the selected points in Sweden

Location	Latitude	Longitude	Height (m)	ξ (arc second)	η (arc second)	$\delta\xi$ (arc second)
Kebnekaise	67.93°N	18.60°E	1,702.3	-10.0	5.0	0.20
Umeå	63.68°N	19.78°E	84.0	-11.1	11.6	0.01
Mårtsbo	60.595143°N	17.258525°E	32.1	-0.6	9.9	0.005
Skövde	57.95°N	14.50°E	262	7.4	5.1	0.04

Note: $\delta\xi$ = effect of the curvature of the plumb line on ξ (cf. Vanicek and Krakiwsky 1986, p. 506).

Effect of DOV on horizontal distance $|\delta_{\text{HD}}|$ in mm (zenith angle= 85°)



Effect of DOV on horizontal distance $|\delta_{\text{HD}}|$ in mm (zenith angle= 85°)

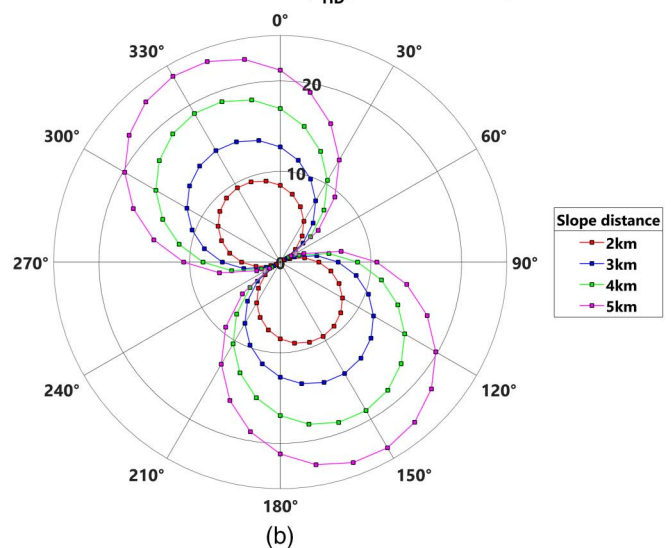


Fig. 7. (a and b) Impact of the DOVs on the horizontal distances considering zenith or vertical angle equal to 85° , different baseline lengths, and azimuth angles using the SWEN17 model in Kebnekaise (the polar plots show max absolute value of $|\delta_{\text{HD}}|$). Unit: mm.

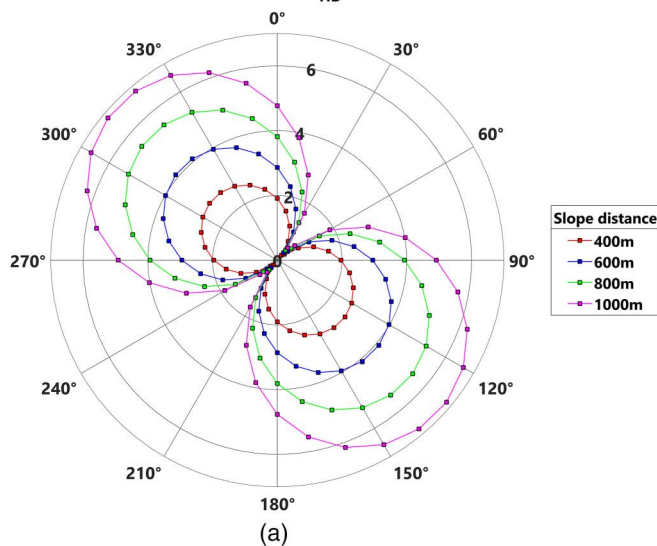
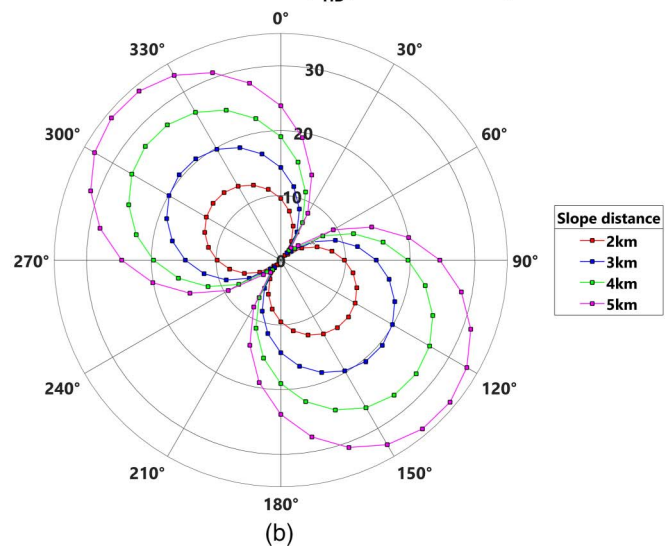
Effect of DOV on horizontal distance $|\delta_{HD}|$ in mm (zenith angle=85°)Effect of DOV on horizontal distance $|\delta_{HD}|$ in mm (zenith angle=85°)

Fig. 8. (a and b) Impact of the DOVs on the horizontal distances considering zenith or vertical angle equal to 85°, different baseline lengths, and azimuth angles using the SWEN17 model in Umeå (the polar plots show max absolute value of $|\delta_{HD}|$). Unit: mm.

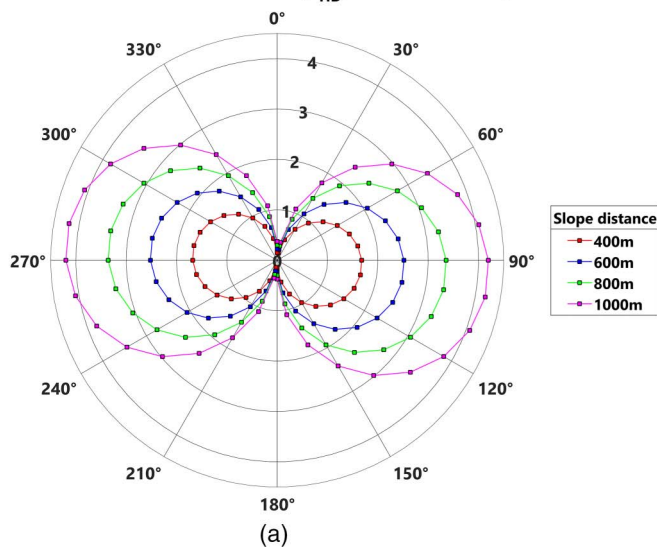
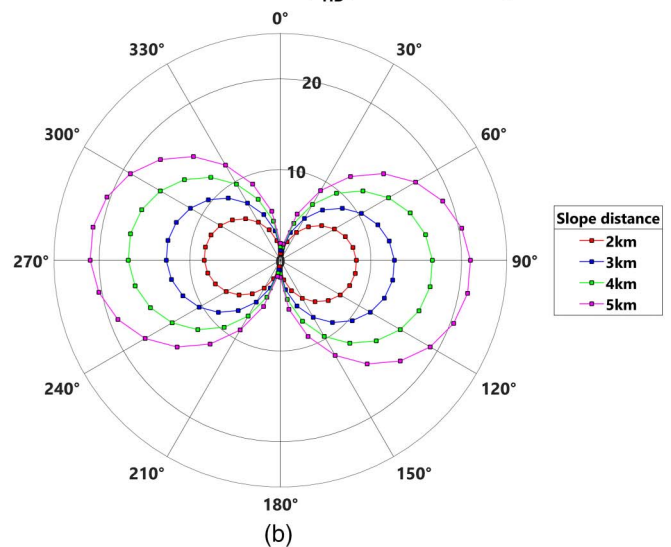
Effect of DOV on horizontal distance $|\delta_{HD}|$ in mm (zenith angle=85°)Effect of DOV on horizontal distance $|\delta_{HD}|$ in mm (zenith angle=85°)

Fig. 9. (a and b) Impact of the DOVs on the horizontal distances considering zenith or vertical angle equal to 85°, different baseline lengths, and azimuth angles using the SWEN17 model in Mårtsbo (the polar plots show max absolute value of $|\delta_{HD}|$). Unit: mm.

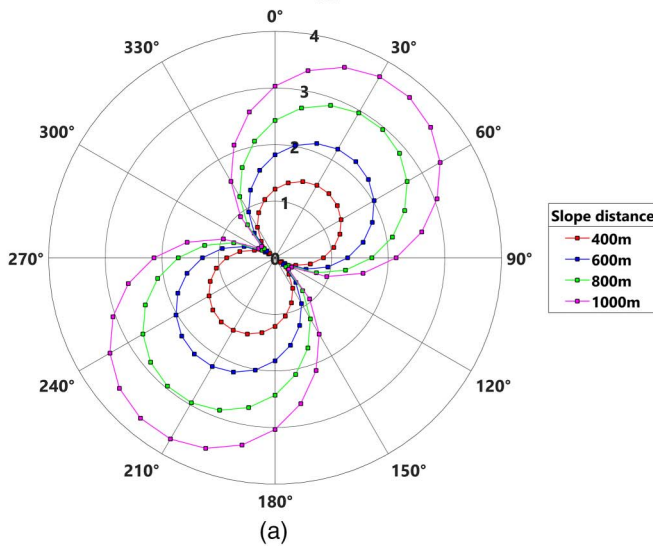
angle in the same stations. This extreme zenith angle value might occur when establishing a control network around high towers and landslide monitoring projects where target points are located above or under the horizontal plane with a large zenith angle value such as 70°.

The statistics of the impact of the DOV on horizontal distances (δ_{HD}) can be seen in Tables S2–S5 in the Supplemental Materials assuming different baseline lengths (between 0.4 and 5 km), azimuths (between 0° and 360°), and two zenith angles (70° and 85°). For instance, considering 400 m baseline length, the absolute values of the DOV effect on the horizontal distance observation were 10.7 (for $Z = 70^\circ$) and 2.7 mm (for $Z = 85^\circ$) in Umeå. Similarly, these values were 6.6 (for $Z = 70^\circ$) and 1.7 mm (for $Z = 85^\circ$) in

Mårtsbo. These values increase rapidly with the length of the slope distance. The values show that the influence of the DOV is significant and considerable. Moreover, studying the specifications of many brands of precise geodetic instruments suitable for control networks [e.g., Leica TM50 (Bern, Switzerland) total station and similar instruments] shows that the distance nominal accuracy is 0.6 mm + 1 ppm (using a round prism). Therefore, any systematic error larger than instrument nominal accuracy should be taken into account before geodetic control network adjustment if the vertical angles are not measured reciprocally.

In addition, the results confirm that the effect of the DOV is also significant in specific directions (azimuths). In other words, the azimuth of the baseline is an influential parameter. With a closer look,

Effect of DOV on horizontal distance $|\delta_{HD}|$ in mm (zenith angle=85°)



Effect of DOV on horizontal distance $|\delta_{HD}|$ in mm (zenith angle=85°)

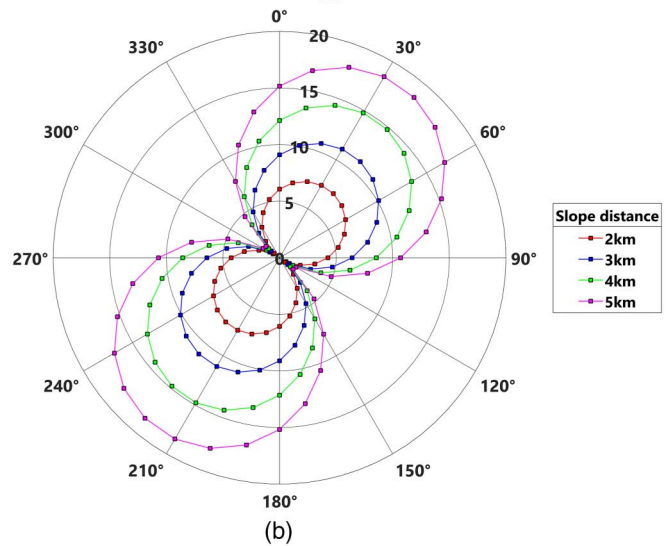


Fig. 10. (a and b) Impact of the DOVs on the horizontal distances considering zenith or vertical angle equal to 85°, different baseline lengths, and azimuth angles using the SWEN17 model in Skövde (the polar plots show max absolute value of $|\delta_{HD}|$). Unit: mm.

the DOV effect will reach its maximum value (approximately) in azimuth equal to 150° and 140° in Kebnekaise and Umeå, respectively, for the applied baselines (Figs. 7 and 8). Similarly, the maximum effects can be seen (approximately) in azimuths equal to 100° and 40° in Mårtsbo and Skövde, respectively (Figs. 9 and 10). Moreover, the DOV impact depends on the azimuth of the baselines. The need for vertical angle measurements to establish geodetic networks, which suffer from the DOV effect and vertical refraction (or their simultaneous reciprocal measurements, which significantly increase the fieldwork and costs of the network campaign), is avoided by using the proposed solution by Shirazian et al. (2021) with the so-called network-aided reductions of slope distance observations. They presented an approach that considered a special geodetic observation strategy to significantly reduce the volume of operations of a precise geodetic network by omitting all measured vertical angles in the network. It changed the design concept for such geodetic networks because the network-aided method decreases data collection time and cost while keeping or even increasing the quality of control networks because it avoids the DOV problem in question (Shirazian et al. 2021). Otherwise, one should use the gravity field corrections (i.e., corrections for the effects of the DOV and geoidal undulations), which need a precise geoid model and gravity and elevation data. We emphasize again that DOV correction must be applied if one uses vertical angles for the reduction of slope distances.

Fig. 11 compares the forward and backward effects of the DOV, considering different distances and assuming a vertical angle equal to 85° in the selected stations. The figure shows the maximum absolute value of δ_{HD} assuming azimuths 150°, 140°, 100°, and 30° for Kebnekaise, Umeå, Mårtsbo, and Skövde, respectively (in the directions that the maximum DOV effects occur). The impact of the DOV on the forward and backward measurements (reciprocal reading) is almost the same if the height difference between the start points and endpoints are small. It confirms that reciprocal reading can be a solution to eliminate the DOV effect when the height difference in the baseline is not large. However, simultaneously, reciprocal reading will increase the fieldwork and costs of the projects (as already mentioned), and it is not recommended. The results show that the differences are ignorable up to 3,000 m baseline

length if both points are in the same elevation. For example, the differences are more significant in Kebnekaise stations due to the large height difference between the start points and endpoints; see more details in Tables S6–S9 in the Supplemental Materials. For example, Table S6 shows the positions of the Kebnekaise station and the endpoints. The position of the baselines' endpoints (varying between 0.4 and 5 km) was calculated using the Vincenty algorithm. Using the obtained position of the endpoints, one can calculate the DOV effect on the assumed vertical angle from the endpoint toward the Kebnekaise station (backward) and finally δ_{HD} . The last column in Tables S6–S9 compares the forward and backward values.

Geometric Effect on Zenith Angles

The geometric effect on the zenith (vertical) angles and consequently on the slope distance reduction is assessed in this section. First, we present the results based on the ellipsoidal Earth model, and finally, the results will be compared with those derived from the spherical assumption. Using Eq. (6), one can calculate the impact of this error by determining forward and backward slope angles from Eqs. (7a) and (7b). We simulated the impact of this error by assuming a point at latitude 60° and longitude 15° and different baseline lengths (100 to 5,000 m) and height differences, varying between 0 and 500 m in Sweden. The geodetic latitude, longitude, and height are first converted into the Cartesian coordinates X , Y , and Z and then into the *enu* system (see equations in Hofmann-Wellenhof et al. 2007, pp. 280–282, for the coordinate transformation). The GRS80 reference ellipsoid (Moritz 2000) was considered for this simulation. Using the obtained coordinates in the *enu* local geodetic coordinate system, one can calculate the reciprocal slope angles and finally δ_{HD}^{C-S} , as explained already. Fig. 12 shows the angles between normals for different baseline lengths. The results show that the curvature-skewness error is significant, and it should be considered for the baselines in the geodetic control networks (see the effect of this error on the slope distance reduction in Table 3 and Fig. S6). It varies between 3.2" (for 100 m baseline length) and 161.6" (for 5,000 m baseline length). According to guidelines (e.g., USACE 2002), geodetic network designers always

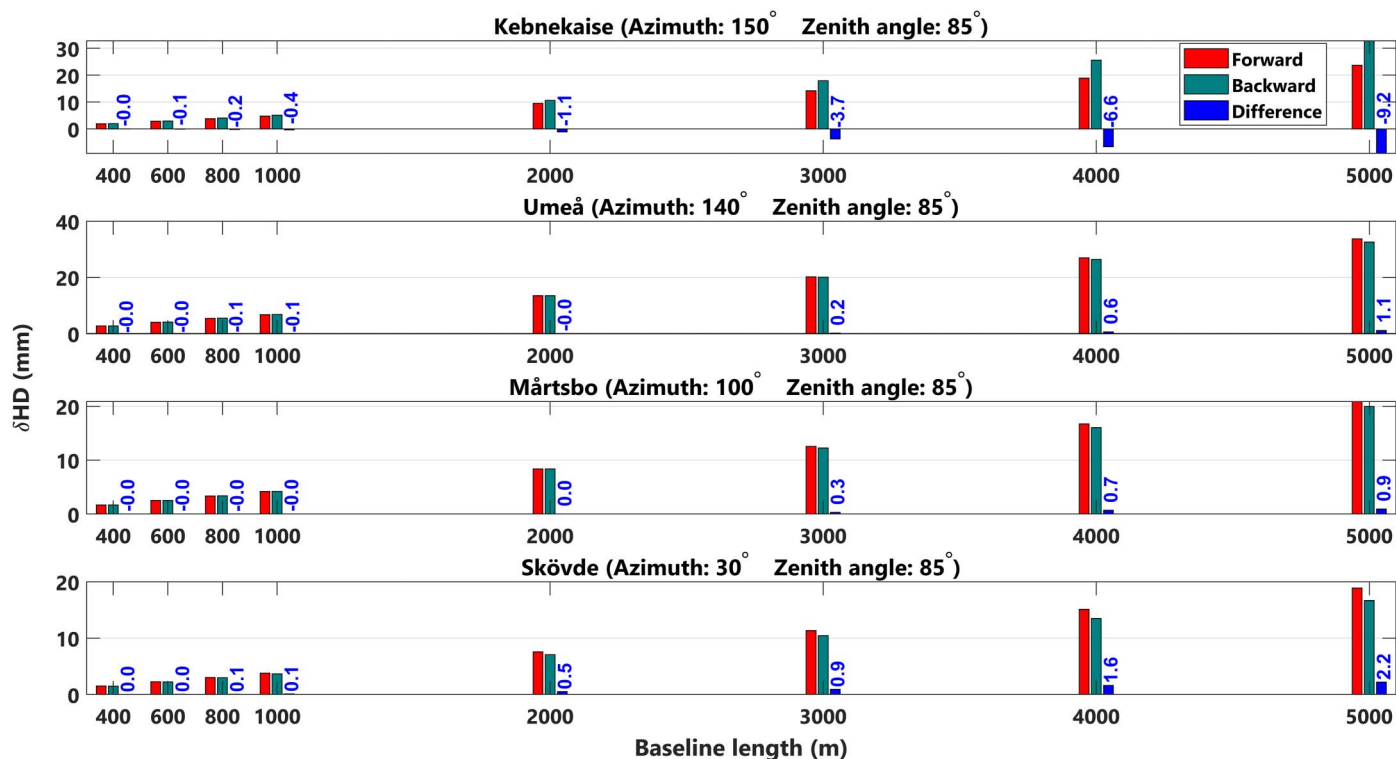


Fig. 11. Forward and backward DOV effect on the horizontal distances assuming different baseline lengths and considering zenith or vertical angle 85°.

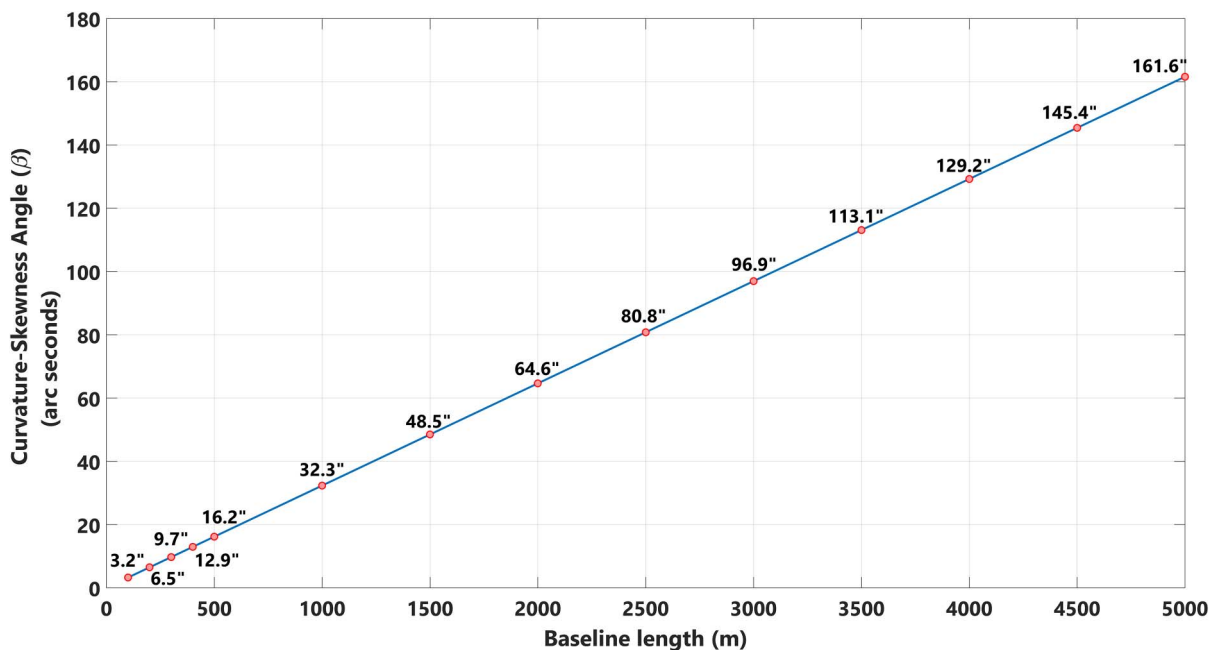


Fig. 12. Angle between normals (β) assuming different baseline lengths and 100 m height difference.

try to design network stations at the same elevation. Therefore, the curvature-skewness effect on the slope distance reduction will be eliminated by designing a proper geodetic network following the guidelines. However, it is not easy to follow the recommendations because of the study areas' situations (e.g., existing rough topography). The curvature-skewness problem does not depend on the

azimuth, and its effect is ignorable (<0.01 mm) for short baselines (see Fig. S7 in Supplemental Materials). We mean to quantify the systematic effects affecting the slope distance reduction to the horizontal distance. For this purpose, we have to create *enu* coordinate systems at the endpoints of the baselines. Through a simulation, we created baselines for which all the station coordinates (geodetic

Table 3. The curvature-skewness effect on slope distance reduction in reciprocal measurements. Unit: mm

Height difference (m)	Distances (m)											
	100	300	500	1,000	1,500	2,000	2,500	3,000	3,500	4,000	4,500	5,000
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.2	0.4	0.4
5	0.0	0.1	0.2	0.4	0.6	0.8	1.0	1.1	1.3	1.4	1.5	1.6
10	0.1	0.2	0.4	0.8	1.2	1.5	1.9	2.3	2.6	2.9	3.2	3.5
50	0.4	1.2	1.9	3.9	5.9	7.8	9.4	11.7	13.6	15.5	17.3	19.2
100	0.8	2.4	3.9	7.8	11.7	15.6	19.5	23.4	27.3	31.1	34.9	38.8
150	1.2	3.5	5.9	11.8	17.6	23.5	29.3	35.2	41.0	46.8	52.6	58.4
200	1.6	4.7	7.8	15.7	23.5	31.3	39.1	46.9	54.7	62.5	70.2	77.9
250	2.0	5.9	9.8	19.5	29.4	39.1	48.9	58.7	68.4	78.1	87.8	97.5
300	—	—	11.7	23.5	35.2	47.0	58.7	70.4	82.1	93.8	105.5	117.1
400	—	—	15.7	31.3	47.0	62.6	78.3	93.9	109.5	125.1	140.7	156.3
500	—	—	19.6	39.2	58.7	78.3	97.9	117.4	136.9	156.4	175.9	195.4

Note: “—” mean that the height differences are unrealistic for the assumed short baselines.

XYZ or BLH) were computed accurately with the exact (simulated) geodesic lengths and azimuths between the start points and endpoints. However, in real practice, we propose the method by Shirazian et al., which establishes a single *enu* system at one of the network points (the closest to the center of the network), measuring its accurate BLH coordinates (by GNSS receiver, for instance).

Table 3 shows the curvature-skewness effect (δ_{HD}^{C-S}) on the slope distance reduction in reciprocal measurements for different baseline lengths and height differences. It can be seen from the table that the curvature-skewness effect is below 0.4 mm (assuming 1 km baseline length) if the geodetic network stations are almost at the same level (for $\Delta H \leq 5$ m). This correction will be notable when one uses a very precise electronic distance meter (EDM), such as a Kern ME500 Mekometer with $0.2 \text{ mm} \pm 0.2 \text{ mm/km}$ range accuracy (Bell 1992), and for large height differences. In other words, the effect of the skewness error is small for zero height differences. Thus, the use of the proposed method is recommended in the case of using highly accurate instruments and avoiding the spherical approximation in the computations.

Fig. S6 visualizes Table 3 and shows the correction for different lengths and height differences. From Fig. S6 and Table 3, one can infer that the correction δ_{HD}^{C-S} increases as the height difference and baseline lengths increase. However, for specific baseline length, larger height differences amplify the effect of the curvature-skewness on the slope distance reduction. Hence, the curvature-skewness correction should be considered for the measured vertical angles to convert the slope to horizontal distances correctly (for large height differences), even if one measures the vertical angles reciprocally. Therefore, the obtained results show that both physical and geometric effects (i.e., the DOV and curvature-skewness problem) will be eliminated by designing the geodetic control points so that the stations are all nearly at the same heights. However, it will generate two further problems. First, as we mentioned already, the refraction error will increase by designing the network at the same elevation. The vertical angles will be very sensitive to atmospheric refraction for vertical angles equal to 90° (Vanicek and Krakiwsky 1986, p. 366). Therefore, there is a recommendation to observe the vertical angles under stable atmosphere conditions: at night or during the winter when there is a thermal inversion. Second, the normal matrix in a three-dimensional (3D) multilateration network becomes ill conditioned if the stations are all nearly at the same heights, which is possible for a geodetic control network. The ill-conditioned problem for the 3D multilateration network was emphasized by Meyer and Elaksher (2021). Therefore, both problems due to designing the geodetic network at the same elevation

highlight the study done by Shirazian et al. (2021). Their proposed method is independent of vertical angles for reducing the slope to horizontal distances.

Furthermore, we compared the curvature-skewness error (using an ellipsoidal model) with the curvature error (spherical model) in this study. Similar results were obtained based on using the spherical assumption [Eq. (8e)]. For example, the geometric errors on the slope distance reduction were 38.8 (ellipsoidal model) and 39.5 mm (spherical model), assuming a 5-km baseline length and 100-m height difference (the difference between the two models was 0.7 mm). We presented both spherical and ellipsoidal models in this study. These can be useful, especially for applying rigorous formulas and avoiding using the spherical approximation. The results show that the spherical assumption is sufficient in all practical cases.

Remark: As can be seen in the results, the geometric effect in the slope distance reduction increases with baseline length (see Fig. S6 and Table 3), whereas the DOV effect does not change so much (Fig. 11). In other words, the forward and backward DOV effects on the horizontal distances (Fig. 11) are approximately equal, but there is a curvature-skewness error on normals. This means that the verticals (perpendicular lines to the geoid) must also be skewed (to keep an equal DOV effect in forward and backward stations), and their angles are close to the one on the reference ellipsoid, that is, the curvature-skewness angle. This issue will be important when one establishes a local astronomy coordinate system. Some nonexperts believe that considering a local coordinate system can overcome physical and geometric errors. This assumption is incorrect because the plumb lines at adjacent points are not parallel. To verify this issue, we conducted a test on real data, and the results and description are presented in the next section.

Curvature-Skewness of the Verticals Effects on Slope Distance Reduction

Quite like normals, verticals (to the geoid) might not coincide at a point and then be skewed. The angles between the verticals must be accounted for when using vertical angles to reduce the slope distances to horizontal ones. One needs to have an accurate geoid model to compute the vertical curvature-skewness angle and its corresponding correction on the vertical angles to attain accurate horizontal distances from slope distances. To show this, we selected four areas in Sweden (Kebnekaise, Umeå, Skövde, and Mårtsbo) and computed the geoid heights (N) for $0.01^\circ \times 0.02^\circ$ grids ($\Delta\varphi = 0.01^\circ$, $\Delta\lambda = 0.02^\circ$) from the SWEN17 model at each grid point in each area. The reader can find the data of the test areas in the Supplemental Materials (see Fig. S8 and Table S10). Then, we

Table 4. Semiaxes of the fitted ellipsoids

Area	Ellipsoid semiaxes (in meters)		
	<i>a</i>	<i>b</i>	<i>c</i>
Kebnekaise	6,378,079.181	6,377,108.551	6,356,815.258
Umeå	6,377,953.394	6,376,673.306	6,356,860.952
Skövde	6,378,384.921	6,377,859.304	6,356,711.342
Mårtsbo	6,378,232.578	6,377,126.821	6,356,784.871

fitted a surface to the 3D grid in each area. According to our results, the best-fitting surface was a triaxial ellipsoid for all areas. There were 35 grid points in each area, of which 30 points were used for surface fitting, and the rest were used to validate the fitted surface. A geoid height at a GNSS levelling point (at which both geodetic height $h = 211.9217$ m and orthometric height $H = 180.2623$ m were available) in the Skövde area ($\varphi = 57.97749308^\circ$, $\lambda = 14.43111579^\circ$) was also used for this validation. The maximum discrepancy between the fitted surfaces and the validation points was about 2×10^{-8} [i.e., the misclosure error of Eq. (9)], which shows the goodness of fit. In Table 4, the fitted ellipsoid specifications are listed.

As known from geometry, the equation of a triaxial ellipsoid reads

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \quad (9)$$

and the normal line to the ellipsoid at an arbitrary point $P(x_0, y_0, z_0)$ can be found from

$$\vec{X} = \vec{\nabla} f(x_0, y_0, z_0) = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right) \bigg|_{x_0, y_0, z_0} \quad (10)$$

It is well known in the literature that the normal line to the geoid is called a *vertical*. Then, we call the curvature-skewness angles of these lines the vertical skewness. This angle between the verticals at Points P_1 and P_2 then reads

$$\alpha = \cos^{-1} \left(\frac{\vec{X}_1 \cdot \vec{X}_2}{\|\vec{X}_1\| \cdot \|\vec{X}_2\|} \right) \quad (11)$$

In Table 5, the curvature-skewness angles between some points in Kebnekaise are listed (for the rest of the areas, please see the Supplemental Materials, Tables S11–S13).

It is clear from Table 5 that it is not possible to ignore the effect of the curvature-skewness angle (even if the vertical angles are observed reciprocally). This means that the plumb lines at adjacent

points are not parallel, even in small-scale geodetic networks. Therefore, one should always be aware of not relying upon the vertical axis of a theodolite to establish a valid up-component of the coordinate system everywhere in the network. This issue is more complex because a local geoid takes sharp changes. Therefore, vertical angle observation is best avoided, and methods like that proposed by Shirazian et al. (2021) should be used as the observation and planning strategy.

Conclusions

We studied the physical and geometric impact on the vertical angle, which is an important observation to convert slope distances to horizontal ones in geodetic networks. The deflection of the vertical and geometric correction should be treated in the data reduction step correctly. One practical solution (according to surveying guidelines) to eliminate these problems is collecting the vertical angles reciprocally and designing geodetic networks so that the station elevations are as much at the same level as possible. Nevertheless, following the guidelines is sometimes difficult because of project circumstances (e.g., establishing a geodetic network for monitoring high towers and structures) and configuration of the geodetic control network due to existing rough topography. Therefore, designing a geodetic network with all stations at the same elevation is not always possible.

The reduction of slope distances requires information about the DOV and the curvature-skewness angle for proper reduction of the baselines in geodetic control networks. Ignoring these effects may lead to significant errors, especially if the height differences between the points are large. Our results show that the DOV and geometric effects on the measured vertical angles are significant even if one measures the vertical angles reciprocally. The DOV effect depends on different factors such as baseline length, azimuth, and height difference in the baselines. We tested the DOV effect in Sweden, specifically at four points (i.e., Kebnekaise, Umeå, Mårtsbo, and Skövde). For example, the results show that the maximum DOV impact on the horizontal distances varied between 1.9 (for 400 m baseline length) and 23.6 mm (for 5,000 m baseline length) considering an 85° zenith angle in Kebnekaise. Similar results were obtained in Umeå, 2.7 and 34.1 mm. However, these values will be enlarged by increasing the length of the slope distances quickly. In addition, the results show that the geometric effect on horizontal distances increases as the height difference and baseline lengths increase. For example, assuming a 10-m height difference in the baseline, the geometric errors obtained were 0.8 (for 1,000 m baseline length) and 2.3 mm (for 3,000 m baseline length), which are larger than the geodetic instruments' distance nominal accuracy. An almost analogous effect exists when a local-astronomy coordinate system (of which the up-component is perpendicular to the geoid) is established in a network. One practical solution to avoid these mentioned problems is applying the method (Shirazian et al. 2021) of the so-called network-aided reduction of slope distance observations so that one can eliminate measuring the vertical angle in geodetic networks.

Data Availability Statement

Some or all data, models, or code that support the findings of this study are available from the corresponding author upon reasonable request. The available data are: (1) the deflection of vertical (DOV) components in Sweden and (2) MATLAB code for calculating the curvature-skewness problem.

Table 5. The curvature-skewness angle in Kebnekaise

Baseline length (m)	Vertical skewness (d°, m', s'')
837.88	0, 0, 27.03
1,115.36	0, 0, 36.00
1,675.76	0, 0, 54.05
2,013.30	0, 1, 04.96
2,513.64	0, 1, 21.08
2,790.46	0, 1, 30.06
3,346.07	0, 1, 48.01
3,532.92	0, 1, 53.97
4,027.21	0, 2, 09.94
4,539.55	0, 2, 26.53
5,121.86	0, 2, 45.31

Acknowledgments

The authors are grateful to Dr. Mahmoud Mohammad Karim from K. N. Toosi University of Technology (Iran), whose lectures inspired the main subject of this paper. We would like to thank the anonymous reviewers for taking the time, their careful reading of our manuscript, and their many insightful comments and suggestions.

Supplemental Materials

Eqs. (S1)–(S9), Figs. S1–S8, and Tables S1–S13 are available online in the ASCE Library (www.ascelibrary.org).

References

- Ågren, J., C. Kempe, and L. Jivall. 2018. "Noggrann höjdbestämning med den nya nationella geoidmodellen SWEN17_RH2000." [In Swedish.] In *Proc., Conf. on Kartdagarna*. Gävle, Sweden: Lantmäteriet.
- Ashkenazi, V., and P. D. Howard. 1984. "An empirical method for refraction modelling in trigonometrical heighting." *Surv. Rev.* 27 (213): 311–322. <https://doi.org/10.1179/sre.1984.27.213.311>.
- Bell, B. 1992. "ME5000 operation." In *Proc., the use and calibration of the Kern ME5000 Mekometer*. Menlo Park, CA: Stanford Linear Accelerator Center, Stanford Univ. Stanford.
- Bomford, G. 1971. *Geodesy*. Oxford, UK: Clarendon Press.
- Bowring, B. R. 1983. "The geodesic inverse problem." *Bull. Géodésique* 57 (1): 109–120. <https://doi.org/10.1007/BF02520917>.
- Brunner, F. K. 1984. *Geodetic refraction*, 216. Berlin: Springer.
- Cronstrand, S. A. 1811. "Stockholms observatorii pol-högd bestämd." *Kongliga Svenska Vetenskaps-Akademiens Handlingar* 32: 291–295.
- Dodson, A. H., and M. Zaher. 1985. "Refraction effects on vertical angle measurements." *Surv. Rev.* 28 (217): 169–183. <https://doi.org/10.1179/sre.1985.28.217.169>.
- Ekman, M., and J. Ågren. 2010. *Reanalysing astronomical coordinates of old fundamental observatories using satellite positioning and deflections of the vertical*. Åland Islands, Sweden: Summer Institute for Historical Geophysics.
- Featherstone, W. E., and J. M. Rüeger. 2000. "The importance of using deviations of the vertical for the reduction of survey data to a geocentric datum." *Aust. Surveyor* 45 (2): 46–61. <https://doi.org/10.1080/00050354.2000.10558815>.
- Grafarend, E. W., and F. Sansò. 2012. *Optimization and design of geodetic networks*. Cham, Switzerland: Springer.
- Heiskanen, W. A., and H. Moritz. 1967. *Physical geodesy*. San Francisco: W. H. Freeman and Company.
- Hirt, C., and G. Seeber. 2002. "Astrogeodätische Lotabweichungsbestimmung mit dem digitalen Zenitkamerasystem TZK2-D." *ZfV-Z. Geodäsie Geoinf. Landmanagement* 127 (1): 388–396.
- Hofmann-Wellenhof, B., H. Lichtenegger, and E. Wasle. 2007. *GNSS—Global navigation satellite systems: GPS, GLONASS, Galileo, and more*. Hoboken, NJ: Springer.
- Jekeli, C. 1999. "An analysis of vertical deflections derived from high-degree spherical harmonic models." *J. Geod.* 73 (1): 10–22. <https://doi.org/10.1007/s001900050213>.
- Krakiwsky, E. J., and D. B. Thomson. 1974. *Geodetic position computations*. Fredericton, NB: Univ. of New Brunswick.
- Kuang, S. 1996. *Geodetic network analysis and optimal design: Concepts and applications*. Lansing, MI: Ann Arbor Press.
- Lantmäteriet. 2020. *Produktbeskrivning, GSD-Höjddata grid 50+ nh*. Gävle, Sweden: Lantmäteriet.
- Meyer, T. H., and A. F. Elaksher. 2021. "Solving the multilateration problem without iteration." *Geomatics* 1 (3): 324–334. <https://doi.org/10.3390/geomatics1030018>.
- Molodenskij, M. S., V. F. Eremeev, and M. I. Yurkina. 1962. *Methods for study of the external gravitational field and figure of the Earth*. [In Russian.] Jerusalem, Israel: Israel Program for Scientific Translations.
- Moritz, H. 2000. "Geodetic reference system 1980." *J. Geod.* 74 (1): 128–133. <https://doi.org/10.1007/s001900050278>.
- Pavlis, N. K., S. A. Holmes, S. C. Kenyon, and J. K. Factor. 2012. "The development and evaluation of the Earth Gravitational Model 2008 (EGM2008)." *J. Geophys. Res. Solid Earth* 117 (4): 1–38. <https://doi.org/10.1029/2011JB008916>.
- Rapp, R. H. 1991. *Geometric geodesy part I*. Columbus, OH: Ohio State Univ.
- Rapp, R. H. 1993. *Geometric geodesy part 2*. Columbus, OH: Ohio State Univ.
- Rollins, C. M., and T. H. Meyer. 2019. "Four methods for low-distortion projections." *J. Surv. Eng.* 145 (4): 04019017. [https://doi.org/10.1061/\(ASCE\)SU.1943-5428.0000295](https://doi.org/10.1061/(ASCE)SU.1943-5428.0000295).
- Schaffrin, B. 1985. "Aspects of network design." In *Optimization and design of geodetic networks*, 548–597. Berlin: Springer.
- Schofield, W., and M. Breach. 2007. *Engineering surveying*. Boca Raton, FL: CRC Press.
- Selander, N. H. 1835. "Undersökning om Stockholms observatorii polhöjd." *Kongliga Svenska Vetenskaps-Akademiens Handlingar* 170–204.
- Shirazian, M., M. Bagherbandi, and H. Karimi. 2021. "Network-aided reduction of slope distances in small-scale geodetic control networks." *J. Surv. Eng.* 147 (4): 04021024. [https://doi.org/10.1061/\(ASCE\)SU.1943-5428.0000375](https://doi.org/10.1061/(ASCE)SU.1943-5428.0000375).
- Sjöberg, L. E., and M. Bagherbandi. 2017. *Gravity inversion and integration*. Cham, Switzerland: Springer.
- USACE. 2002. *Engineering and design: Structural deformation surveying*. Washington, DC: USACE.
- Vanicek, P., and E. J. Krakiwsky. 1986. *Geodesy: The concepts*. Amsterdam, Netherlands: Elsevier.
- Vincenty, T. 1975. "Direct and inverse solutions of geodesics on the ellipsoid with application of nested equations." *Surv. Rev.* 23 (176): 88–93. <https://doi.org/10.1179/sre.1975.23.176.88>.
- Wargentin, P. 1759. "Stockholms observatorii pol-högd bestämd." *Kongliga Svenska Vetenskaps-Akademiens Handlingar* 20: 305–314.