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The Theory of Joining-Systems

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ABSTRACT. The theory of joining-systems (TJS), as developed in this chapter, consists of three main parts, developed after the informal introduction and overview in Sections 1 and 2. One part (Section 3) is the abstract theory of joining-systems, providing the framework for the subsequent analysis. Two other parts introduce those concepts and results of the theory that are in focus for the representation of normative systems. The first of these parts (Section 4) presents the model of condition implication structures (cis's) as applied to well-known issues in legal theory. In the second part (Section 5), the cis model of TJS is applied to a comprehensive new field, namely the theory of "intervenients". In a developed normative system, intervenient concepts serve as vehicles of inference for going from ultimate descriptive grounds to ultimate deontic consequences. Among the issues dealt with are: Boolean compounds of intervenients, intervenients as organic wholes, narrowing or widening of intervenients, the typology of various kinds of intervenient minimality.

Handbook of Deontic Logic and Normative Systems, 545–634.

1 The field of research and its origins

In the analysis of normative systems, one of the approaches is to represent a normative system as a *deductive mechanism*, giving a normative output for an input of facts. In modern literature, the foremost origin of this approach is the work *Normative Systems* by the Argentinians Carlos E. Alchourrón and Eugenio Bulygin. To this tradition belongs as well the recent "input-output logic" by David Makinson and Leon van der Torre and the Theory of Joining-Systems (TJS) proposed by the present authors.

A theory of representation for normative systems will be incomplete unless attention is paid to the role of *intermediate concepts* within the system (for example, the role of legal concepts such as ownership). If a normative system is represented as a deductive mechanism, there will be an emphasis on the role of intermediate concepts as "vehicles of inference" within the system. In this respect, the origin of later developments comes from Scandinavian legal philosophy in the 1950's, in particular the work of Anders Wedberg and Alf Ross.

1.1 Cases and solutions in the theory of Alchourrón and Bulygin

Alchourrón and Bulygin introduce the idea of deductive mechanism by contrasting the Aristotelian conception of science with the idea of deductive system in modern theory [Alchourrón and Bulygin, 1971, pp. 43ff.]. The notion of deductive system is based on Tarski's notion of deductive consequence, satisfying the following four requirements [Alchourrón and Bulygin, 1971, pp. 48ff.]:

1. The set of the consequences of a set of sentences consists solely of sentences.
2. Every sentence belonging to a given set is to be regarded as a consequence of this set.
3. The consequences of the consequences are, in turn, consequences.
4. If a sentence of a conditional form $(y \supset z)$ is a consequence of the set of sentences X, then z is a consequence of the set of sentences resulting from adding to X the sentence y.

Adopting the Tarskian conception of deductive system, Alchourrón and Bulygin conceive of a normative system as a set of sentences deductively correlating pairs of sentences. A set α of sentences deductively correlates a pair $\langle p, q\rangle$ of sentences if q is a deductive consequence of $\{p\} \cup \alpha$, or, using the relation Cn of consequence, if $q \in Cn(\{p\} \cup \alpha)$. Moreover, the statement $q \in Cn(\{p\} \cup \alpha)$ is equivalent to $(p \supset q) \in Cn(\alpha)$ where $\supset$ is the symbol for truth-functional implication [Alchourrón and Bulygin, 1971, pp. 54ff.]

For a set α to be a normative system the additional requirement is made that there be at least one pair $\langle p, q\rangle$ where $q \in Cn(\{p\} \cup \alpha)$ such that p is a "case" and q is a "solution". A solution is a normative sentence expressed in terms of a descriptive sentence (deontic content) preceded by a deontic operator for command, prohibition or permission. So, the character of the system as normative depends on the deontic character of the solutions inferred in the system. In the words of [Alchourrón and Bulygin, 1971, p. 169]: "Justifying the deontic qualification of an action by means of a normative system consists in showing that the obligation, the prohibition or the permission of this action can be inferred from (i.e., is a consequence of) this system."

If propositional logic is used as a basis, it is usually presupposed that p, q are closed sentences with no free variables, i.e., for example, p is the sentence "Smith has promised to pay Jones \$100" and q is "Smith has an obligation to pay \$100 to Jones". In these sentences, individuals are referred to by individual constants (names). While it is true that a normative system may correlate sentences of this kind, a set of sentences containing individual names is not, however, an appropriate representation of a normative system. A normative system expresses general rules where no individual names occur. If the task is to represent a normative system this feature of generality has to be taken into account.

When Alchourrón and Bulygin speak of normative "solutions" being correlated to "cases", however, they have in mind correlation of "generic" cases to "generic" solutions. They emphasize the distinction between individual and generic cases, and an analogous distinction holds for solutions. An individual case is a situation or a state of affairs. As such, appropriately, it should be described by a closed sentence. On the other hand, a generic case is a property or a set of individual cases, defined by a property.[1] Therefore, a "case" in the generic sense relevant to Alchourrón and Bulygin is an object described by an open sentence. It can be argued that, when the expression $q \in Cn(\{p\} \cup \alpha)$ is said to express that α correlates q to p, q and p must be thought of as "open" sentences (like "x has promised to pay \$$y$ to z", "It shall be that x pays \$$y$ to z"), not prefixed by any universal quantifier.[2]

1.2 Input-output logic

In a series of papers, Makinson and van der Torre have developed a logic called "input-output logic", see for example [Makinson and van der Torre, 2000; Makinson and van der Torre, 2003]. If G is a generating set, then $x \in out(G, A)$, i.e., x belongs to the output of A under G, if and only if $(A, x) \in out(G)$. The principal *out*-operation in input-output logic does not require reflexivity or contraposition.

Input-output logic can, but need not, apply specifically to normative systems, where norms are represented as ordered pairs.[3] The construction of norms in input-output logic, however, is different from the construction in [Alchourrón and Bulygin, 1971]. In Alchourrón and Bulygin, if a is a case and x is a solution, it is assumed that x is a normative sentence (a solution, see above). In contrast, in input-output logic, a generating set G of ordered pairs $\langle a, x\rangle$ can be understood as a set of conditional obligations in spite of the fact that x, the consequence, is descriptive rather than normative. The normative character, in this case, depends on the specific character of the set G as a set of conditional obligations. (Similarly if $\langle a, x\rangle$ is a conditional permission.)

For further details, the reader is referred to the Chapter "Input/output logic" of the present Handbook. A remark on the interrelation between

[1] By an individual case is meant an element of the universe of discourse. See [Alchourrón and Bulygin, 1971, p. 28, and p. 10]. A generic case is described alternatively as a subset of the universe of discourse, defined by a property, or as this defining property itself. See [Alchourrón and Bulygin, 1971, p. 29].

[2] Cf. [Alchourrón and Bulygin, 1971, p. 49], and the comments in [Lindahl and Odelstad, 2004, sect. 1.1].

[3] [Makinson and van der Torre, 2000, p. 383 and p. 392].

input-output logic and TJS is given below, Section 6.2.2.

1.3 The theory of joining-systems TJS

In TJS, implications are seen as relations between two objects. Thus a statement "a implies b" expresses that an implicative relation holds from a to b. The specific character of the objects a and b is a matter of which model is chosen for the abstract theory.

A first view of TJS is as follows. A simple normative system contains three basic kinds of implicative relations:

- a relation R_1 over a set A_1 of grounds,
- a relation R_2 over a set A_2 of consequences,
- a relation J from the grounds in A_1 to the consequences in A_2 (expressing the norms of the system).

We note that, though each of R_1, R_2 and J is a binary implicative relation, the relation J is different in kind from R_1, and R_2. Thus while the point of the latter two relations is to order elements of A_1 and A_2, respectively, relation J is a "correspondence", with the purpose of assigning consequences in A_2 to grounds in A_1 and vice versa. (This is particularly perspicuous in the case where A_1 and A_2 are disjunct.)

A picture of a joining relation is shown in Figure 1.

The resulting structures or systems are: The structure $\mathcal{A}_1 = \langle A_1, R_1 \rangle$ of grounds, the structure $\mathcal{A}_2 = \langle A_2, R_2 \rangle$ of consequences, and the system $\langle \mathcal{A}_1, \mathcal{A}_2, J \rangle$, called a joining-system, where the elements of J are joinings from $\mathcal{A}_1$ to $\mathcal{A}_2$. (The elements of the joining relation J constitute a subset of $A_1 \times A_2$, representing the norms of the normative system.) For a joining-system $\langle \mathcal{A}_1, \mathcal{A}_2, J \rangle$, if $\langle a_1, a_2 \rangle \in J$ (where $a_1 \in A_1$ and $a_2 \in A_2$), we say that a_1 is a ground for a_2 and a_2 is a consequence of a_1.

To the three relations R_1, R_2, J will be added a fourth implicative ordering relation $\trianglelefteq$, called "narrowness", over the set of elements in J. These elements (i.e., the norms from $A_1 \times A_2$) can be more or less "narrow", and this is expressed by the relation $\trianglelefteq$. From another aspect, $\trianglelefteq$ expresses implication between the norms in J. Thus, the expression $\langle a_1, a_2 \rangle \trianglelefteq \langle b_1, b_2 \rangle$ means that $\langle a_1, a_2 \rangle$ is at least as narrow as $\langle b_1, b_2 \rangle$, and also that $\langle a_1, a_2 \rangle$ implies $\langle b_1, b_2 \rangle$.

1.4 TJS for simple normative systems

TJS has a wider range of application than the representation of normative systems. As will appear in Sections 2 and 3, the general theory of joining-systems can be applied to quasi-orderings of any kind. Within this range, a

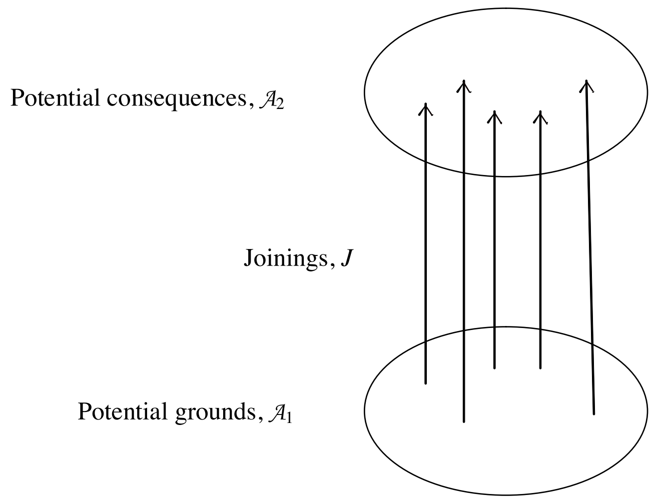

Figure 1

field of special interest is that of what may be called "Many-sorted implicative conceptual systems" (cf. [Odelstad, 2008]). From the perspective to be adopted here, a special area of this kind is the representation of normative systems with conditional norms. In TJS, this problem is dealt with in terms of joinings of normative consequences in A_2 to grounds in A_1.

If the sentence "a implies b" expresses a (conditional) norm, it is assumed that b, the consequence, is normative. In this respect, the representation of norms in TJS is akin to the theory of correlation of normative solutions to cases in the work of Alchourrón and Bulygin, but different from the representation of norms in input-output logic. The specific character of various normative consequences in TJS is dealt with in terms of so-called normative positions, made up by a combination of deontic concepts (constructed by "Shall", "May" for obligation and permission) and action concepts (constructed from "x sees to it that ...").

1.5 Normative positions in TJS

An important refinement of classical deontic logic is the theory of normative positions as the combination of a standard deontic operator *Shall*, expressing command (or *May*, expressing permission) with an action operator *Do* ("Do(x,F)" for "x sees to it that F"), and exploiting the possibilities of external and internal negation of sentences where these operators are combined. See Chapter "The theory of normative positions " in the present Handbook.

As an illustration, imagine a normative system $\mathcal{N} = \langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ such that $\langle a_1, a_2, \rangle \in J$. Suppose F is the condition that the police is informed of which political party x sympathizes with. Let a_1, a_2 be as follows:

a_1 : x is not suspected of any crime, and y is a police authority.

a_2 the conjunction of (1)-(6) below:

(1) May Do(x,F)

(2) May Do(x, $\neg F$),

(3) May ($\neg$Do(x,F) & $\neg$Do(x,$\neg F$))

(4) $\neg$May Do(y,F) (= Shall $\neg$ Do(y,F))

(5) May Do(y, $\neg F$),

(6) May ($\neg$Do(y,F) & $\neg$Do(y,$\neg F$))

Among these, (1)-(3), (5-6) express permissions, while (4) expresses a prohibition. (1) expresses that x may see to it that the police is informed of which political party x sympathizes with, (2) that x may see to it that the police is not so informed, (3) that x may be passive in this respect. (4) expresses that it shall be the case that y (a police authority) does not see to it that the police is informed, and so on. As will appear later, the conjunction of (1)-(3) exemplifies one-agent type T_1 of normative positions while the conjunction of (4)-(6) exemplifies one-agent type T_4.

As will be developed in Section 4.4 below, the TJS version of normative positions combines the TJS approach to joining-systems with an explicitly algebraic model of the theory of normative positions. In the system of grounds and consequences of a normative system, the algebraic version of normative positions is an algebra of normative consequences intended to handle the stratum $\mathcal{A}_2$ of a normative joining-system $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$. In Section 4.4.1, we introduce an example of conditional norms concerning the normative positions of the owners of two adjacent estates.

1.6 Subtraction and addition of norms in TJS

An important issue within the representation of normative systems is the handling of changes, in the sense of subtracting and/or adding norms to the system. Section 4.3 below provides an example showing how TJS deals with these issues in terms of the lattice-like structure of so-called minimal joinings. The example concerns the legal effects of an illegal transfer of goods belonging to someone else. We illustrate the transition from an original normative system $\mathcal{S}_I$, satisfying specific requirements for minimal joinings, via an unsatisfactory system $\mathcal{S}_{II}$, to systems $\mathcal{S}_{III}$ and $\mathcal{S}_{IV}$, once more satisfying the requirements for joining-systems.

1.7 Intermediaries and intervenients

1.7.1 Facts, normative positions and intermediaries

Legal rules attach obligations, rights, normative positions to facts, i.e., the occurrence of actions and events, or the presence of circumstances. Normative positions are, so we might say, legal consequences of these facts. Facts and normative positions are objects of two different sorts; we might call them Is-objects and Ought-objects. In a legal system, when Ought-objects are said to be "attached to" or to be "consequences of" Is-objects, there is sense of direction. In a legal system, inferences and arguments go from Is-objects to Ought-objects, not vice versa.

In the Is-Ought partition, something very essential is missing, namely the great bulk of more specific legal concepts. A few examples are: property, tort, contract, trust, possession, guardianship, matrimony, citizenship, crime, responsibility, punishment. These concepts are links between grounds on the left hand side and normative consequences on the right hand side of the scheme below:

Facts	*Links*	*Normative positions*
Events	Ownership	Obligations
Actions	Valid contract	Claims
Circumstances	Citizenship (etc.)	Powers (etc.)

Using this three-column scheme, we might say that ownership, valid contract, citizenship etc. are attached to certain facts, and that normative positions, in turn, are attached to these legal positions.

As an example, Amendment XIV, Section 1, of the Constitution of the United States reads as follows:

> "All persons born or naturalized in the United States, and subject to the jurisdiction thereof, are citizens of the United States and of the State wherein they reside. No State shall make or enforce any law which shall abridge the privileges or immunities of citizens of the United States; nor shall any State deprive any person of life, liberty, or property, without due process of law; nor deny to any person within its jurisdiction the equal protection of the laws."

Two central terms in this constitutional rule are "citizen" and "person". The rule enumerates grounds for being a citizen of the United States and pronounces a number of legal consequences, expressed in terms of "shall", of this condition. It does not assert any grounds for being a "person", but it pronounces a number of legal consequences attached to personhood. Within the U.S. constitutional system, the article just referred to is supplemented by other rules established by the Constitution and by constitutional court decisions. These rules together, by specifying grounds and consequences, indicate the role of the term "citizen" or "person" within the system.

1.7.2 Wedberg and Ross on vehicles of inference

In the 1950's, each of the two Scandinavians Wedberg and Ross proposed the idea that a legal term such as "ownership", or "x is the owner of y at time t" is a syntactical tool serving the purpose of economy of expression of a set of legal rules.[4]

As an example, the function of the term "ownership" is illustrated as follows by [Ross, 1951], cf. [Ross, 1956 and 1957]:

$$\left.\begin{array}{l} F_1 \rightarrow \\ F_2 \rightarrow \\ F_3 \rightarrow \\ \vdots \\ F_p \rightarrow \end{array}\right\} \quad O \rightarrow \quad \left\{\begin{array}{l} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{array}\right.$$

Figure 2

In the picture, the letters are to be interpreted as follows:

- $F_1 - F_p$ for: x has lawfully purchased y, x has inherited y, x has acquired y by prescription, and so on.
- $C_1 - C_n$ for: judgment for recovery shall be given in favor of x against other persons retaining y in their possession, judgment for damages shall be given in favor of x against other persons who culpably damage y, if x has raised a loan from z that it is not repaid at the proper time, z shall be given judgment for satisfaction out of y, and so on.

The letter "O" is a link between the left hand side and the right hand side. It can be read "x is the owner of y".

In Ross's scheme, the number of implications to ownership from the grounds for ownership is p (since the grounds are $F_1, ..., F_p$); similarly the

[4]In the same year 1951, when Ross published his well-known essay "Tû-Tû" in a Danish Festschrift [Ross, 1951] (English translation [Ross, 1956 and 1957]), Wedberg published an essay on the same theme in the Swedish journal *Theoria* [Wedberg, 1951]. Possibly, the two authors arrived at these ideas independently of each other. Cf. [Wedberg, 1951, p. 266, n. 15], and [Ross, 1956 and 1957, p. 822, n. 6].

number of implications from ownership to consequences of ownership is n (since there are n consequences). Therefore, the total number of implications in the scheme is $p + n$. On the other hand, if the rules were formulated by attaching each C_j among the consequences to each F_i among the grounds, the number of rules would be $p \cdot n$. Consequently, by the formulation in the scheme, the number of rules is reduced from $p \cdot n$ to $p + n$, a number that can be much smaller [Wedberg, 1951, pp. 273f.]. In this way, economy of expression is obtained.[5] (Cf., however, below, Section 1.7.4, on reductionism.)

1.7.3 Intermediaries and meaning

Both Wedberg and Ross emphasize that intermediaries like "ownership" fulfil their deductive purpose even if they are not defined. Ross claims that "ownership" is a meaningless word in legal language:

> "... the 'ownership' inserted between the conditioning facts and the conditioned consequences is in reality a meaningless word, a word without any semantic reference whatever, serving solely as a means of presentation." [Ross, 1956 and 1957, p. 820]

Already in 1944 (in a lecture in Uppsala), Anders Wedberg proposed the idea that the concept of a "right", as it appears in a normative system, is a syntactical tool for inferences, not a concept with "independent meaning".

> "In the normative rules, the concepts of rights function as syntactical tools, not as concepts with independent meaning." (See [Lindahl, 2004, p. 189, n. 16] for the reference.)

In his essay in 1951, [Wedberg, 1951], Wedberg, more cautiously, proposes this as a "third alternative", beside the alternatives of defining ownership in terms of grounds or in terms of consequences, respectively (alternatives one and two).

A plausible interpretation of Wedberg's idea of "not independent meaning" is that the rules stating the grounds and consequences of ownership (cf. Ross's figure above) are meaningful and that the sentence "O is the property of P at t" has a purposeful role as a component of these rules but that it has no meaning in abstraction from the rules where it functions as a vehicle of inference.

[5]The similarities between Wedberg's and Ross' ideas are striking. Both use the example of ownership. Central ideas propounded by both of them are: By use of the linking term, the number $p \cdot n$ of rules is reduced to $p + n$, and, the linking term has no independent meaning (Wedberg) or has no semantical reference (Ross).

> "It may be shocking to unsophisticated common sense to admit such 'meaningless' expressions in the serious discourse of legal scientists. But, as a matter of fact, there is no reason why all expressions employed in a discourse, which as a whole is highly 'meaningful', should themselves have a 'meaning'." [Wedberg, 1951, p. 273]

[Sartor, 2009] contrasts the idea of vehicles of inference with the idea of legal concepts as "categories" in a domain ontology.[6] In the latter perspective, meaning inheres in words or terms, and the meaning of sentences results from the meaning of their lexical components. (See [Sartor, 2009, pp. 236f.]. In jurisprudential writing, systematization is sometimes achieved by the ordering of legal concepts in conceptual trees or pyramids.[7] (As a well-known analogue from natural science, we may think of the Linnaean system of plants, which influenced eighteenth century conceptual jurisprudence in Germany.) If such an ordering is to be congruent with an existing normative system, however, it should accord with the role the concepts have as vehicles of inference within the system. If A and B are subcategories of category C, then category C indicates some properties which members of A and B have in common.[8] As regards concepts in a normative system, these common properties may regard either grounds or consequences or both, according to the rules of the system in view.

Since there are many legal systems, there are (to take an example) many concepts of ownership, more or less similar. Thus one concept of ownership is ownership as a vehicle of inference in Swedish private law on January 1st 2010. This concept of ownership is determined by the particular normative system referred to; consequently, the concept is replaced by another whenever the grounds or consequences of ownership in the system are changed. We note that, when several different concepts (for example, ownership in actual Swedish law and ownership in Anglo-Saxon common law) are called "concepts of ownership", it is suggested that these varieties have properties in common, justifying that they are called "concepts of ownership". In particular, the concepts in view can have a common historical origin, and the "institution" that they are used for expressing (the institution of ownership) can have the same social purpose or function in the different systems.

[6] An earlier version of Sartor's paper is [Sartor, 2007].

[7] Cf. [Lindahl, 2000], in particular pp. 166f., on the reasoning of the German eighteenth century jurist Georg Friedrich Puchta. A systematization of concepts appears as well in the arrangement of norms in civil codes such as the German *Bürgerliches Gesetzbuch* and the French *Code Civil*.

[8] A recent development is the idea of semantic networks and inheritance, see [Horty *et al.*, 1990], (referred to by [Sartor, 2009, p. 243, n. 27]. The focus in [Horty *et al.*, 1990] is on defeasibility, in this case "multiple inheritance with exceptions".

Considerations of this kind are relevant for a critical assessment of the ownership rules of particular normative systems, and may cause assessment of what is the "essential content" of ownership.[9]

1.7.4 Reductionism

In the Ross-Wedberg example on ownership, the set of legal rules illustrated by the picture can be reformulated in two rules:

(1) $(F_1 \vee ... \vee F_m) \to O$.

(2) $O \to (S_1 \wedge ... \wedge S_n)$.

If the middle term M is eliminated, we get the single rule:

(3) $(F_1 \vee ... \vee F_m) \to (S_1 \wedge ... \wedge S_n)$.

The most economical way to express the rules of the two arrays above would seem to be by a single sentence like (3). By reductionism regarding intermediaries is meant the idea that legal reasoning might in general proceed directly from facts to normative consequences so as to dispense with intermediate concepts.

Concerning the accomplishment of reduction, two complications have to be born in mind. Firstly, the bulk of so-called "legal" concepts are intermediaries, and these intermediaries constitute complex networks. (Cf. [Lindahl and Odelstad, 2011]) Secondly, many legal intermediaries are vague or "open textured", so that power to decide on grounds and consequences for the intermediaries is conferred on judges and other persons who apply the law (see below, Section 5.2.2).

The question whether, in principle, it is possible to do away with the intermediaries is complex and will not be answered here. A formal theory for handling intermediaries, however, is needed both for any attempt to eliminate them and for representing the system as it is without reduction.

1.7.5 Open legal concepts

As mentioned, there are numerous cases where legal concepts are vague or "open textured", and power to interpret the concepts is conferred on judges and other persons who apply the law. Obvious examples are such concepts as "negligent" or "reasonable" but considerable openness also is a feature of such concepts as "public interest", "contract" and "ownership".

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[9] To exemplify, in German constitutional law there is a guarantee of protection for the "essential content" (*Wesensgehalt*) of the basic rights of the German Constitution. In an essay by the Swedish philosopher Ingemar Hedenius, Max Weber's idea of "ideal types" is applied to the concept of ownership, where normative systems are represented as different alternatives of fulfilment on each of several dimensions. (See [Hedenius, 1977, pp. 130-55].) According to Hedenius' proposal, the concept of ownership in particular normative systems can be critically assessed according to their degree of fulfilment on the dimensions introduced.

An example might be the legal rule stipulating the ground for what, in Swedish law, is called "having a relationship similar to being married". If two persons are not married, nevertheless they can have a relationship similar to being married. From such a condition particular legal consequences follow by the law. First, if the relationship is dissolved, property acquired by one of the parties for use in common shall be partitioned between the parties according to rules similar to those applied when a marriage is dissolved. Secondly, if the relationship of the parties is dissolved, their dwelling can be allotted to that party who needs it most.

The law does not specify exactly which facts give rise to a "relationship similar to being married".[10] However, there are a number of criteria. Let us consider the following eleven criteria, calling them $F_1, F_2, ..., F_{11}$:

> F_1 : cohabiting, F_2 : housekeeping in common, F_3 : having children in common, F_4 : having sexual intercourse, F_5 : having confirmed the relation by a contract, F_6 : living in emotional fellowship, F_7 : being faithful, F_8 : giving mutual support, F_9 : sharing economic assets and debts, F_{10} : having no legal impediments to marriage, F_{11} : having no similar relationship to another person.

If all of the criteria are satisfied by persons i and j, their relationship is "similar to being married". Conversely, if none of them is satisfied, their relationship is not "similar to being married". These two rules belong to established law.

However, the law does not say what is the result if some of the conditions are satisfied while others are not. This means that, in a sense, the set of grounds for having a relationship similar to being married is "open", and the grounds are not specified completely.

A great amount of legal concepts are "ground-open" like "relationship similar to being married". When such a concept occurs in a legal argument, there is room and need for decisions to be made by courts and other authorities applying the law. This task is an obstacle to reductionist efforts to do away with legal intermediaries in favor of rules attaching deontic consequences directly to factual events, actions, circumstances. In legal argument from facts to deontic consequences, the argument is a sequence of steps, passing through a number of stations involving legal concepts. Insofar as the concepts are open, decisions have to be made step by step.

[10] In 2003, a new statute (SFS 2003: 376) on cohabitant partners ("sambor") was enacted in Sweden. In article 1, paragraph 1, there is a definition of "cohabitant partners", intended to be a little more precise: "By cohabitant partners is meant two persons who live together permanently in a partner relationship and have their housekeeping in common." (Translated here.)

"Relationship similar to being married" is a concept that is ground-open, in the sense we have indicated. Similarly, a legal concept can be consequence-open. Taking a concept like "ownership", "citizenship" or "matrimony", for some deontic consequences it is established that they do follow, for others it is established that they do not follow. However, there are as well consequences for which it is not established whether they follow or not. Then the concept is consequence-open.

"Being the owner of" can serve as an example of a concept that is to some extent consequence-open. Thus it need not, for example, be entirely settled to what extent and by what means the owner of an estate may exclude others from entering on his/her ground.

The phenomenon of open concepts in a normative system is connected with the limits on what can be achieved by a legislator. If a legislator attempts to avoid openness, the probability increases that the norms enacted become oversimplified. As clearly understood already by Aristotle, it is not possible to create a complete legal code of "established law" without incurring into error by oversimplification:

> ".. all law is universal but about some things it is not possible to make a universal statement which shall be correct. In those cases, then, in which it is necessary to speak universally, but not possible to do so correctly, the law takes the usual case, though it is not ignorant of the possibility of error. And it is none the less correct; for the error is [not] in the law nor in the legislator but in the nature of the thing, since the matter of practical affairs is of this kind from the start. When the law speaks universally, then, and a case arises on it which is not covered by the universal statement, then it is right, where the legislator fails us and has erred by oversimplicity, to correct the omission - to say what the legislator himself would have said had he been present, and would have put into his law if he had known. Hence the equitable is just, and better than one kind of justice - not better than absolute justice but better than the error that arises from the absoluteness of the statement. And this is the nature of the equitable, a correction of law where it is defective owing to its universality. In fact this is the reason why all things are not determined by law, that about some things it is impossible to lay down a law, so that a decree is needed. For when the thing is indefinite the rule also is indefinite, like the leaden rule used in making the Lesbian moulding; the rule adapts itself to the shape of the stone and is not rigid, and so too the decree is adapted to the facts." [Aristotle, Nicomachean

Ethics, EN 1137b]

The issue of open legal concepts will be dealt with in Section 5.2.2 below.

1.7.6 Intermediaries outside the realm of legal systems

The idea of intermediaries is applicable outside the realm of legal systems. An example is Dummett's theory of language. Dummett distinguishes between the conditions for applying a term and the consequences of its application. According to Dummett both are parts of the meaning. Dummett exemplifies by the use of the term "Boche" as a pejorative term Cf. [Kremer, 1988; Lindahl and Odelstad, 2006a; Lindahl and Odelstad, 2008a; Sartor, 2007; Sartor, 2009]. (Since the example is interesting from a philosophical point of view, we use it even though it has the disagreeable feature of being offensive to German nationals.)

> "The condition for applying the term to someone is that he is of German nationality; the consequences of its application are that he is barbarous and more prone to cruelty than other Europeans. We should envisage the minimal joinings in both directions as sufficiently tight as to be involved in the very meaning of the word: neither could be severed without altering its meaning. Someone who rejects the word does so because he does not want to permit a transition from the grounds for applying the term to the consequences of doing so. The addition of the term 'Boche' to a language which did not previously contain it would produce a non-conservative extension, i.e., one in which certain statements which did not contain the term were inferable from other statements not containing it which were not previously inferable." [Dummett, 1973, p. 454]

Dummett's example illustrates how the use of a word is determined by two rules (1) and (2):

(1) Rule linking a concept a to an intermediary m : If $a(x, y)$ then $m(x, y)$,

(2) Rule linking intermediary m to a concept b : If $m(x, y)$ then $b(x, y)$.

The rules (1) and (2) can be compared to the rules of introduction and rules of elimination, respectively, in Gentzen's theory of natural deduction in [Gentzen, 1934]. If this comparison is made, (1) is regarded as an introduction rule and (2) as an elimination rule for m. (See [Lindahl and Odelstad, 2008a, sect. 1.2.3].)

In natural science, the idea of "intermediate" has been applied to the term "force" within physical theory. As is observed by [Wedberg, 1982, pp. 11ff.]

during the eighteenth century several thinkers thought of the forces spoken of in mechanics as a kind of mathematical fictions, useful for describing the movements of bodies in a convenient way. What exists in physical reality, according to this view, are configurations of mass, speeds, and accelerations. Forces are fictions, but they enable us to describe the interrelations of the former entities in a compact way. As Wedberg mentions, Berkeley is among the thinkers who held this opinion.

The position, held by Berkeley and others, that "force" is merely a device for compact expression, closely resembles the idea of intermediaries. This resemblance becomes even more obvious if the position in view is described in Wedberg's own words:

> "If a body k with mass m is in a particular (spatial and temporal) relation to certain other bodies, we say that a force of magnitude f affects k. If a force of magnitude f affects k, then k receives an acceleration a satisfying the equation:
>
> (i) $f = a \cdot m$
>
> Thus the force occurs as a middle term in the pair of hypothetical statements:
>
> (ii) Given a certain configuration of mass, a certain force exists.
>
> (iii) Given a certain force, a certain acceleration results.
>
> If the middle term is eliminated, we arrive at the conclusion:
>
> (iv) Given a certain configuration of mass, a certain acceleration results." [Wedberg, 1982, p. 11]

An objection to Berkeley's idea that forces are "fictions", however, is raised by Wedberg in pointing out that the term "force" can be defined in terms of such entities that Berkeley considers as real. Such a definition, in Wedberg's words, might be formulated as a definition of the entire statement (see [Wedberg, 1982, p. 12]):

> The body k exerts a force f upon the body k'.

A definition of this statement, then, can read as follows:

> f is the product of the acceleration a, which k' receives from k and the mass of k'.

In connection with the possibility of defining "force" in terms of "real" entities, we recall the possibility of defining ownership, either in terms of grounds or in terms of consequences (Wedberg's "first" and "second" alternatives).

Another interesting example from physics is found in the work of Henri Poincaré. Poincaré proposed that "gravitation" can be regarded as an intermediary (*un intermédiaire*). According to Poincaré, the proposition "the stars obey Newton's laws" can be broken up into two others, namely (1) "gravitation obeys Newton's laws" and (2) "gravitation is the only force acting on the stars". Among these, proposition (1) is a definition and not subject to the test of experiment, while (2) is subject to such a test. "Gravitation", according to Poincaré, is an intermediary. Poincaré maintains that in science, when there is a relation between two facts A and B, an intermediary C is often introduced by the formulation of one relationship between A and C, and another between C and B. The relation between A and C, then, is often elevated to a principle, not subject to revision, while the relation between C and B is a law, subject to such revision. See [Poincaré, 1907, pp. 124f.], in the chapter "Is science artificial?" On the analogous question of definition and norm in a normative system, cf. [Lindahl, 1997, p. 298].

Still another example concerns probability (see [Lindahl and Odelstad, 1999a]). Consider statements of the kind "the probability of the event A equals m" (where m is a real number). Using the notion of conditions, introduced below in Section 4.2, page 596, one may speak of conditions on events, for example the condition of having probability m. Such a condition can be regarded as an intermediary between two conceptual structures, one concerning frequencies and symmetries, and the other concerning how one ought to choose between different games. It is a plausible idea that the so-called objective, or frequency, interpretation of probability deals with the structure of grounds for probability conditions, whereas the so-called subjective interpretation deals with the structure of consequences. This suggestion seems to assign a proper role to each of the two interpretations.

For a treatment of intermediate concepts in connection with weighing of interests in urban planning, see [Odelstad, 2002; Odelstad, 2009].

1.7.7 Counts-as-theory

When a rule r of a legal system $\mathcal{N}$ attaches an intermediary m, e.g., "x and y have made a contract to the effect that z", to a conjunction a of facts, the rule r can be expressed in different ways, e.g. "if a then m", or, sometimes, "*a counts as m*". A logical analysis of sentences of the kind "x counts-as y in s", where s is an institution (s can be a normative system), was proposed in [Jones and Sergot, 1996; Jones and Sergot, 1997].[11] The work of Jones and

[11] The original motivation of Jones and Sergot was, so it seems, to give a formal characterization of "institutionalized power", see [Jones and Sergot, 1997, pp. 349ff.]. For a comment on this matter, see [Lindahl and Odelstad, 2008a, sect. 3.5.3, n. 22].

Sergot on "Counts-as" has been continued by a number of other authors, in particular in the book-length study by Davide Grossi [Grossi, 2007]. For further details on Counts-as, the reader is referred to Chapter "Constitutive norms and counts as conditionals" of the present Handbook. A remark on the interrelationship between Counts-as and TJS, see below, Section 6.2.1.

1.7.8 "Intervenient" as a technical notion in TJS

An essential part of the theory of joining-systems is the theory of intervenients. Though this theory aims at providing tools for analyzing intermediaries as they appear in law, language, morals, and so on, "intervenient" is a technical notion defined (see Definition 5.2, below, Section 5.1) at the abstract algebraic level, used as a tool for analyzing different kinds of what, informally, is called intermediaries. The notion of intervenient is tied to the TJS approach, focusing on a normative system as a deductive mechanism and on intermediaries as vehicles of inference. Therefore, in the development of the theory of intervenients, the idea of economy of expression has a central role. This relates both to the effective representation of a normative system by intervenients and to changes in such a system accomplished by changing grounds and/or consequences of intervenients.

Special themes regarding intervenients dealt with in this Chapter are what we call "organic wholes" (Section 5.2.1), open concepts and "narrowing of intervenients" (Section 5.2.2), and the typology of intervenients (Section 5.2.4).

1.8 Advice to readers

Though a substantial part of the chapter is abstract and formal, there are as well several parts that are semi-formal. This holds for next Section 2, which is a first introduction to TJS, as well as for the subsections on *cis* applications in Sections 4 and 5. More exactly, these subsections are: Section 4.3 on subtraction and addition of norms, Section 4.4.1 on ownership to an estate, Section 5.2.1 on organic wholes of intervenients, Section 5.2.2 on open concepts and the "narrowing" of intervenients, and Section 5.2.3 on the legal example of grounds and consequences of ownership and trust.

2 First introduction to TJS

2.1 General TJS irrespective of intervenients

2.1.1 Strata and joining systems

The structure of grounds as well as the structure of consequences will be called a *stratum*. The word "stratum" is understood here in the sense of the result of arrangement of the parts or elements of something.[12] More precisely, in TJS, the general structure of a stratum is a set A of objects, ordered by an implicative relation R, which is binary, reflexive and transitive. It is not assumed that R is antisymmetric, nor that it is not. In other words, a stratum is conceived of as a quasi-ordering $\langle A, R\rangle$ of objects from a set A. (Another term for quasi-ordering is *preordering*.) The relation R is a relation ordering the objects within a stratum, and, therefore, is called an *intrastratum* relation.

In TJS, the relation J is an *interstrata* implicative relation from elements of a stratum of grounds to elements of a stratum of consequences. As will be made more explicit subsequently, the relation J (which, normally, is not a function) provides a "correspondence" between these two strata, depicting the set of grounds on the set of consequences and vice versa. In this respect, relation J differs from relation R which is an intrastratum ordering relation.

As mentioned (see Section 1.3), a *joining-system* $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ consists of two strata $\mathcal{A}_1, \mathcal{A}_2$ and a relation J. TJS leaves room for different kinds of structures over each of $\mathcal{A}_1, \mathcal{A}_2$. For $1 \leq i \leq 2$, a stratum can be a quasi-ordering $\langle A_i, R_i\rangle$, where A_i is (simply) a set, or it can be a "lattice-based quasi-ordering" $\langle L_i, \wedge, \vee, R_i\rangle$, where $\langle L_i, \wedge, \vee\rangle$ is a lattice, or it can be a "Boolean quasi-ordering", $\langle B_i, \wedge, ', R_i\rangle$, where $\langle B_i, \wedge, '\rangle$ is a Boolean algebra. A special case is where, for a lattice-based quasi-ordering $\langle L_i, \wedge, \vee, R_i\rangle$ or a Boolean quasi-ordering $\langle B_i, \wedge, ', R_i\rangle$, R_i is the relation $\leq$ of $\langle L_i, \wedge, \vee\rangle$ or of $\langle B_i, \wedge, '\rangle$, respectively.

As will appear, the definition of "joining-system" is the same, independently of which is the type of the strata connected in the joining-system, only provided that each stratum fulfills the minimum requirement of being a quasi-ordering. Thus while there is flexibility as regards the types of strata, the definition of joining-system gives stability to the theory: As we will see, a joining-system exhibits a number of important properties, relevant for the representation of a normative system.

While both the intrastratum R and the interstrata J express implication, an essential difference between R and J is that between "one-sort" objects and "two-sorts" objects. In TJS, the intrastratum R is a relation between objects conceived of as being of the same sort; in contrast, the interstrata relation J is a relation between objects thought of as being of two sorts. As regards normative systems, the idea of two sorts applies in particular to the difference between empirical/descriptive and normative. (In another area, consider the difference between physical and mental.)

[12]Cf. the online *Free Dictionary*: "One of a number of layers, levels, or divisions in an organized system." Note that "stratum" as used here is not to be understood in the sense of: "one of several parallel layers of material arranged one on top of another."

Norms are represented by ordered pairs $\langle a_1, a_2\rangle$ where a_1, a_2 are of different sorts. The most general version of TJS is where the strata $\mathcal{A}_1, \mathcal{A}_2$ of a joining-system $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ are simply quasi-orderings. A substantial part of TJS will be developed within this general framework. As will appear, in this version, TJS yields a number of results for the formal representation of normative systems. In particular, by the relation $\trianglelefteq$ of narrowness (see above, end of Section 1.3), there is an implicative structure over the norms of the system, and the system can be expressed in an economic way by its set of "minimal joinings".

2.1.2 Minimal joinings

Suppose that a norm $\langle a_1, a_2\rangle$ is a joining from a stratum $\mathcal{A}_1$ of grounds to a stratum $\mathcal{A}_2$ of consequences. Then, if (in a sense to be defined) a_1 is a "weakest ground" for a_2, and a_2 is a "strongest consequence" of a_1, the pair $\langle a_1, a_2\rangle$ represents what in TJS is called a *minimal joining*. If a normative system fulfills a requirement called "connectivity", any norm in the system is always implied by a minimal joining.

In TJS, a normative system can be represented in a convenient way by its set of minimal joinings, and therefore, minimality is decisive for how economy of expression is accomplished and for how changes of a system can be effectively achieved. Furthermore, in a well-structured normative system, the set of minimal joinings has a number of perspicuous structural properties. Thus, firstly, the set of minimal joinings can be ordered in an interesting way as a lattice-like structure. Secondly, if $\langle a_1, a_2\rangle$ belongs to the set J of joinings, let us call the ground a_1 the "bottom" of the joining $\langle a_1, a_2\rangle$ and the consequence a_2 the "top" of this joining. Then, as we will see, there is a similarity between the set $\min J$ of minimal joinings and the set of bottoms of $\min J$ as well as to the set of tops of $\min J$.

2.2 Intervenients in TJS

Suppose that we have in view three joining-systems $\mathcal{S}_1 = \langle \mathcal{A}_1, \mathcal{A}_2, J_{1,2}\rangle$, $\mathcal{S}_2 = \langle \mathcal{A}_2, \mathcal{A}_3, J_{2,3}\rangle$, $\mathcal{S}_3 = \langle \mathcal{A}_1, \mathcal{A}_3, J_{1,3}\rangle$ such that these systems constitute a chain in the sense that by $J_{1,2}$ you can go from $\mathcal{A}_1$ to $\mathcal{A}_2$, by $J_{2,3}$ you can go from $\mathcal{A}_2$ to $\mathcal{A}_3$, and by $J_{1,3}$ (using relative product) you can go directly from $\mathcal{A}_1$ to $\mathcal{A}_3$. In a sense, the stratum $\mathcal{A}_2$ is intermediate between $\mathcal{A}_1$ and $\mathcal{A}_3$. Certain elements in A_2 can be *intervenients* between elements in A_1 and elements in A_3.[13] (See Figure 3 on page 565.) If $a_1 \in A_1$, and $a_2 \in A_2$ and $a_3 \in A_3$, a_2 *corresponds* to the pair $\langle a_1, a_3\rangle$ if, in a sense to be defined, later, a_1 is the weakest ground in $\mathcal{A}_1$ for a_2 and a_3 is the strongest

[13] Note that we use calligraphic letters $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ for the quasi-orderings $\langle A_1, R_1\rangle$, $\langle A_2, R_2\rangle$, $\langle A_3, R_3\rangle$ and we use italics A_1, A_2, A_3 for the domains of these quasi-orderings.

consequence in $\mathcal{A}_3$ of a_2. The investigation of intervenients following in this

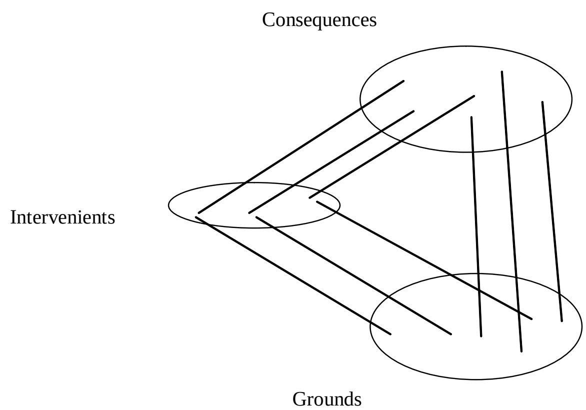


Figure 3

chapter has in view the structure and properties of the intervenients. To this subject-matter belongs a number of special issues. A few examples are as follows. If economy of expression is related to the notion of minimal joinings, what can be said about intervenients and minimality? Is there a typology of intervenients and minimality? Under what conditions can a normative system be represented by a base of intervenients? Furthermore, there is the issue of Boolean operations (conjunction, disjunction, negation) on intervenients. If a_2, b_2 are intervenients from $\mathcal{A}_1$ to $\mathcal{A}_3$, then what can be said about $a_2 \wedge b_2$, $a_2 \vee b_2$ and (the negations) a_2', b_2'? How do Boolean compounds of intervenients relate to corresponding compounds of grounds and of consequences? All of these questions are essential to the formal structure of intervenients and have a direct bearing on the formal representation of intermediaries in a normative system.

2.2.1 Subject-matter of sections 3-5

The following three main Sections 3-5 are organized as follows. (We recall what was said in Section 2.1.1 about joining as a relation between elements of two strata.) In Section 3, the basic theory of joining-systems is developed, while Section 4 is devoted to the theory of different kinds of strata. In Section 3, dealing with joining-systems in general, very little is presupposed about the structure of strata. In Section 4, on the other hand, the character of strata is the subject-matter of more differentiation. Here, what is in view is joining-systems where strata are Boolean-like structures or lattice-like structures. Since the development in Section 4 is intended for the representation of normative systems, the focus there will mainly be on so-called Boolean joining-systems. Section 5 is devoted to the theory of intervenients in Boolean joining-systems.

It should be observed that the general results regarding lattice-like structures in Section 3 are essential for the analysis of joining-systems, including the analysis of Boolean joining-systems (later pursued in Section 4) and the analysis of intervenients (in Section 5).

3 Formal development of TJS

3.1 Basic concepts

Much of the study of ordering relations in mathematics seems to have partial orderings as its basic structure. Lattices and Boolean algebras, for example, are partially ordered sets. In the study of norms and conceptual systems, it is more convenient to take quasi-orderings as the formal framework. The reason for choosing quasi-orderings instead of partial orderings is that in a quasi-ordering $\langle A, R\rangle$ two objects a and b can be similar with respect to R (for example, by having the same extension) without being identical. This feature is useful when dealing with concepts.

In the next subsection (Section 3.1.1), the notion of quasi-ordering is defined. After that, in the subsequent subsubsections, we generalize some well-known mathematical notions, so as to apply to quasi-orderings.

3.1.1 Quasi-orderings

First a note on terminology. Suppose that R is ν-ary relation on a set A and that X is a subset of A. Then $R \cap X^\nu$ is denoted R/X and is called the restriction of R to X.

Definition 3.1 *The binary relation R is a* quasi-ordering on *A if R is transitive and reflexive in A.*

(As mentioned, another name for quasi-ordering is preordering.)

Writing Q for the *equality* part of R we say that xQy holds iff xRy and yRx. Also, writing P for the *strict* part of R we put xPy iff xRy and not yRx.

A quasi-ordering is closely related to a partial ordering. If $\langle A, R\rangle$ is a quasi-ordering and Q is the equivalence part of R, then R generates a partial ordering on the set of Q-equivalence classes generated from A.

Definition 3.2 *Suppose that R is a quasi-ordering on A and that $X \subseteq A$ and $x \in X$. Then,*
(1) *x is a* minimal element *in X with respect to R iff there is no $y \in X$*

such that yPx,
(2) x *is a* maximal element *in* X *with respect to* R *iff there is no* $y \in X$ *such that* xPy.
(3) *The set of minimal elements in* X *with respect to* R *is denoted* $\min_R X$ *and the set of maximal elements of* X *with respect to* R *is denoted* $\max_R X$.
(4) x *is a* least element *in* X *with respect to* R *iff for all* $y \in X$, xRy,
(5) x *is a* greatest element *in* X *with respect to* R *iff for all* $y \in X$, yRx.

Note that in a quasi-ordering $\langle A, R\rangle$, a greatest and a least element in a set $X \subseteq A$ need not be unique. But if x and y are greatest elements (or least elements) in X with respect to R, then xQy.

3.1.2 Quasi-lattices and complete quasi-lattices

As will appear in Section 3.2.2, the notions of least upper bound and greatest lower bound are important in the definition of a joining-system. These notions are usually defined for partial orderings and not for quasi-orderings. Since quasi-ordering is a basic structure in TJS, we generalize the notions of least upper bound and greatest lower bound to quasi-orderings. We use ub and lb as abbreviations for upper bound and lower bound respectively, and lub and glb for least upper bound and greatest lower bound respectively. We note that (in contrast to what holds for partial orderings) a least upper bound or a greatest lower bound relative to a quasi-ordering $\langle A, R\rangle$ need not be unique.

Definition 3.3 *Let* R *be a quasi-ordering on a set* A *with* $X \subseteq A$. *Then*
$\mathrm{ub}_R X = \{a \in A \mid \forall x \in X : xRa\}$
$\mathrm{lb}_R X = \{a \in A \mid \forall x \in X : aRx\}$
$\mathrm{lub}_R X = \{a \in A \mid a \in \mathrm{ub}_R X \ \& \ \forall b \in \mathrm{ub}_R X : aRb\}$
$\mathrm{glb}_R X = \{a \in A \mid a \in \mathrm{lb}_R X \ \& \ \forall b \in \mathrm{lb}_R X : bRa\}$.

According to standard algebraic terminology, a partially ordered set $\langle L, \leq\rangle$ is a lattice if for all $a, b \in L$, $\sup_{\leq}\{a, b\}$ and $\inf_{\leq}\{a, b\}$ exist in L. (In connection with partial orderings, we prefer to use sup and inf instead of lub and glb respectively.) $\langle L, \leq\rangle$ is *complete* if $\inf_{\leq} X$ and $\sup_{\leq} X$ exist for all $X \subseteq L$. We generalize these notions to quasi-orderings.[14]

Definition 3.4 *If* $\langle A, R\rangle$ *is a quasi-ordering such that*

$$\mathrm{lub}_R\{a, b\} \neq \varnothing \textit{ and } \mathrm{glb}_R\{a, b\} \neq \varnothing \textit{ for all } a, b \in A,$$

[14] Note that the concept of completeness for lattices, quasi-lattices, and quasi-orderings should not be confounded with completeness in the sense that an ordering relation R on a set A is called complete if for all $x, y \in A$ it holds that xRy or yRx.

then $\langle A, R\rangle$ *will be called a* quasi-lattice. *If* $\mathrm{lub}_R\, X \neq \varnothing$ *and* $\mathrm{glb}_R\, X \neq \varnothing$ *for all* $X \subseteq A$*, then* $\langle A, R\rangle$ *is a* complete *quasi-lattice.*

If $\langle A, \leq\rangle$ is a partial order then $a \in \sup_{\leq} \varnothing$ iff a is the smallest element in A with respect to $\leq$ and $a \in \inf_{\leq} \varnothing$ iff a is the greatest element in A with respect to $\leq$. (See for example [Grätzer, 2011, p. 5].) Analogously, if $\langle A, R\rangle$ is a quasi-order then

(i) $a \in \mathrm{lub}_R\, \varnothing$ iff a is a smallest element in A with respect to R

(ii) $a \in \mathrm{glb}_R\, \varnothing$ iff a is a greatest element in A with respect to R.

We note that if a quasi-lattice is finite, then it is complete.

Theorem 3.5 *Suppose that* $\langle A, R\rangle$ *is a quasi-lattice, that* Q *is the indifference-part of* R*, and that* A_Q *is the set of* Q*-equivalence classes generated by elements of* A*. Then* $\langle A_Q, R^*\rangle$*, where* $[a]_Q\, R^*\, [b]_Q$ *iff* aRb*, is a lattice. If* $\langle A, R\rangle$ *is a complete quasi-lattice then* $\langle A_Q, R^*\rangle$ *is a complete lattice.*

In analogy with what holds of complete lattices, see [Grätzer, 2011, p. 50], the following holds of a complete quasi-lattice.

Theorem 3.6 *Let* $\langle A, R\rangle$ *be a quasi-ordering in which* $\mathrm{glb}_R\, X \neq \varnothing$ *for all* $X \subseteq A$*. Then* $\langle A, R\rangle$ *is a complete quasi-lattice.*

By duality, the theorem holds if instead $\mathrm{lub}_R\, X \neq \varnothing$ for all $X \subseteq A$.

In lattice theory the notion of a sublattice is introduced. Suppose $\langle L, \leq\rangle$ is a lattice and $\varnothing \neq M \subseteq L$. Let, furthermore, $\leq^* = \leq /M$. Then $\langle M, \leq^*\rangle$ is a sublattice of $\langle L, \leq\rangle$ if $a, b \in M$ implies that $\sup_{\leq^*} \{a, b\} = \sup_{\leq} \{a, b\}$ and $\inf_{\leq^*} \{a, b\} = \inf_{\leq} \{a, b\}$. We now generalize the notion of a sublattice to quasi-lattices and define the notion of a subquasi-lattice.

Definition 3.7 *Suppose that* $\langle A, R\rangle$ *is a quasi-lattice,* $X \subseteq A$ *and* $S = R/X$*. Then* $\langle X, S\rangle$ *is a* subquasi-lattice *of* $\langle A, R\rangle$ *if* $x, y \in X$ *implies that* $\mathrm{lub}_R\, \{x, y\} \supseteq \mathrm{lub}_S\, \{x, y\} \neq \varnothing$ *and* $\mathrm{glb}_R\, \{x, y\} \supseteq \mathrm{glb}_S\, \{x, y\} \neq \varnothing$*.*

Theorem 3.8 *If* $\langle A, R\rangle$ *is a quasi-lattice and* $\langle X, S\rangle$ *a subquasi-lattice of* $\langle A, R\rangle$*, then* $\langle X_Q, S^*\rangle$ *is a sublattice of* $\langle A_Q, R^*\rangle$*.*

(See the notation introduced in Theorem 3.5.)

3.2 Joining-systems

3.2.1 Narrowness

In TJS, the relation of “narrowness” is highly important. It is used in the definition of a joining-system, since it determines the relation of implication between norms and the set of minimal joinings (cf. above Section 2.1.2). The minimal joinings are essential in a normative system, since they serve as the tool for a succinct representation of the system.

Definition 3.9 (1) *The* narrowness relation determined by *the quasi-orderings $\langle A_1, R_1\rangle$ and $\langle A_2, R_2\rangle$ is the binary relation $\trianglelefteq$ on $A_1 \times A_2$ such that $\langle a_1, a_2\rangle \trianglelefteq \langle b_1, b_2\rangle$ iff $b_1 R_1 a_1$ and $a_2 R_2 b_2$.*
(2) *$\langle x_1, x_2\rangle$ is a* minimal element *in $X \subseteq A_1 \times A_2$ with respect to $\langle A_1, R_1\rangle$ and $\langle A_2, R_2\rangle$ if $\langle x_1, x_2\rangle$ is a minimal element in X with respect to $\trianglelefteq$. The set of minimal elements in X with respect to $\trianglelefteq$ is denoted* $\min^{R_2}_{R_1} X$*. (When there is no risk of ambiguity we write just* $\min X$*.)*

Note that $\trianglelefteq$ is a quasi-ordering, i.e. transitive and reflexive. Let $\simeq$ denote the equality part of $\trianglelefteq$ and $\triangleleft$ the strict part of $\trianglelefteq$. Then the following holds:

$\langle a_1, a_2\rangle \simeq \langle b_1, b_2\rangle$ iff $b_1 Q_1 a_1$ & $a_2 Q_2 b_2$
$\langle a_1, a_2\rangle \triangleleft \langle b_1, b_2\rangle$ iff $(b_1 P_1 a_1$ & $a_2 R_2 b_2)$ or $(b_1 R_1 a_1$ & $a_2 P_2 b_2)$

where Q_i is the equality-part of R_i and P_i is the strict part of R_i.

The notion of narrowness is illustrated in Figure 4. Note that $\langle x_1, x_2\rangle$ is

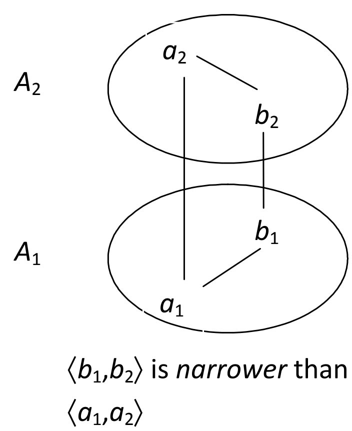


Figure 4

a minimal element in $X \subseteq A_1 \times A_2$ with respect to $\langle A_1, R_1\rangle$ and $\langle A_2, R_2\rangle$ if there is no $\langle y_1, y_2\rangle \in X$ such that $\langle y_1, y_2\rangle \triangleleft \langle x_1, x_2\rangle$, i.e. if there is no element $\langle y_1, y_2\rangle \in X$ such that $x_1 R_1 y_1$ & $y_2 P_2 x_2$, or $x_1 P_1 y_1$ & $x_2 R_2 y_2$.

In TJS, *up-sets* with respect to the narrowness-relation will be of special interest. We give an explicit definition of up-set with respect to the narrowness-relation here.[15]

Definition 3.10 *Suppose that $\mathcal{A}_1 = \langle A_1, R_1\rangle$ and $\mathcal{A}_2 = \langle A_2, R_2\rangle$ are quasi-orderings and $K \subseteq A_1 \times A_2$. Then we say that K is an* up-set *with respect to $\trianglelefteq$ if the following holds: For all $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$, if $\langle a_1, a_2\rangle \in K$ and $\langle a_1, a_2\rangle \trianglelefteq \langle b_1, b_2\rangle$, then $\langle b_1, b_2\rangle \in K$.*

3.2.2 The definition of a joining-system

As mentioned in Section 2.1.1, while TJS is flexible as regards the character of strata $\mathcal{A}_1$ and $\mathcal{A}_2$, in TJS the definition of "joining-system" is the same, independently of which is the type of the strata connected in the joining-system, only provided that each stratum fulfills the minimum requirement of being a quasi-ordering.

The definition of joining-system is as follows.

Definition 3.11 *A* joining-system (Js)*, is an ordered triple $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ such that $\mathcal{A}_1 = \langle A_1, R_1\rangle$ and $\mathcal{A}_2 = \langle A_2, R_2\rangle$ are quasi-orderings, and $J \subseteq A_1 \times A_2$, and the following conditions are satisfied where $\trianglelefteq$ is the narrowness relation determined by $\mathcal{A}_1$ and $\mathcal{A}_2$:*
(1) *for all $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$, if $\langle a_1, a_2\rangle \in J$ and $\langle a_1, a_2\rangle \trianglelefteq \langle b_1, b_2\rangle$, then $\langle b_1, b_2\rangle \in J$,*
(2) *for any $X_1 \subseteq A_1$ and $a_2 \in A_2$, if $\langle a_1, a_2\rangle \in J$ for all $a_1 \in X_1$, then $\langle b_1, a_2\rangle \in J$ for all $b_1 \in \mathrm{lub}_{R_1} X_1$,*
(3) *for any $X_2 \subseteq A_2$ and $a_1 \in A_1$, if $\langle a_1, a_2\rangle \in J$ for all $a_2 \in X_2$, then $\langle a_1, b_2\rangle \in J$ for all $b_2 \in \mathrm{glb}_{R_2} X_2$.*

(In what follows, when we use the expression $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$, we presuppose that $\mathcal{A}_1 = \langle A_1, R_1\rangle$ and $\mathcal{A}_2 = \langle A_2, R_2\rangle$.)

If $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ is a joining-system, then the elements in J are called *joinings* from $\mathcal{A}_1$ to $\mathcal{A}_2$, and we call J the *joining-space* in $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$. We call $\mathcal{A}_1$ the *bottom-structure* and $\mathcal{A}_2$ the *top-structure* in the *Js* $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$.

Requirement (1) in the definition of a joining-system means that the joining-space J is an up-set with respect to the narrowness-relation. Note that from requirement (1) it follows, for example, that if $\mathcal{A}_1, \mathcal{A}_2$ are lattices such that $a_1, b_1 \in A_1$, $a_2, b_2 \in A_2$ and $\langle a_1, a_2\rangle \in J$ then, $\langle a_1 \wedge b_1, a_2\rangle \in J$ and $\langle a_1, a_2 \vee b_2\rangle \in J$.

As an analogy, in propositional logic, for the implicative connective $\rightarrow$ it holds that from the conjunction of $p_1 \rightarrow q_1$ and $p_2 \rightarrow q_2$ it follows that

[15] For the notion of "up-set" in general, see for example [Davey and Priestley, 2002, p. 20].

if $q_1 \to p_2$ then $p_1 \to q_2$. Requirement (1) stipulates a similar result for a combination of the three implicative relations R_1, R_2 and J in a joining-system.

For a joining-system $\langle \mathcal{A}_1, \mathcal{A}_2, J \rangle$ conceived of as representing a normative system, let us interpret a formula $\langle x_1, x_2 \rangle \in J$ so as to mean that $\langle x_1, x_2 \rangle$ is a norm in $\langle \mathcal{A}_1, \mathcal{A}_2, J \rangle$. Then the import of requirement (1) is that if it holds that $\langle a_1, a_2 \rangle$ is a norm in $\langle \mathcal{A}_1, \mathcal{A}_2, J \rangle$ and $b_1 R a_1$ and $a_2 R b_2$ then $\langle b_1, b_2 \rangle$ as well is a norm in $\langle \mathcal{A}_1, \mathcal{A}_2, J \rangle$. This requirement is a corner-stone in the TJS approach to normative systems as deductive mechanisms. In a sense, a normative system $\langle \mathcal{A}_1, \mathcal{A}_2, J \rangle$ is represented by the quasi-ordering $\langle J, \trianglelefteq \rangle$. As we shall see, however, there are other representations that are more economical in expression.

The import of requirements (2) and (3) is easier to see if we suppose that $\langle A_1, R_1 \rangle$ and $\langle A_2, R_2 \rangle$ are lattices so that $\wedge$ and $\vee$ are defined for the elements in A_1 and A_2, respectively. In this case, from requirements (2) and (3) it follows: If $\langle a_1, a_2 \rangle \in J$ and $\langle b_1, a_2 \rangle \in J$ then $\langle a_1 \vee b_1, a_2 \rangle \in J$ (requirement (2)). And if $\langle a_1, a_2 \rangle \in J$ and $\langle a_1, b_2 \rangle \in J$ then $\langle a_1, a_2 \wedge b_2 \rangle \in J$ (requirement (3)).

We note that a joining-system as here defined gives rise to a closure system (see Section 3.2.5 below). Also, we note that in requirement (2) we do not presuppose that $\mathrm{lub}_{R_1} X_1 \neq \varnothing$ and in requirement (3) we do not presuppose that $\mathrm{glb}_{R_2} X_2 \neq \varnothing$. Furthermore note that $\langle \mathcal{A}_1, \mathcal{A}_2, \varnothing \rangle$ and $\langle \mathcal{A}_1, \mathcal{A}_2, A_1 \times A_2 \rangle$ are joining-systems, the *empty* joining-system and the *trivial* joining-system respectively. A joining-system that is not empty or trivial is called a *proper* joining-system.

In the definition of a joining-system, we do not presuppose that the domains in the quasi-orderings are disjunct sets. This is indeed the case in many intended applications, but in a large number of typical applications there is some overlap between the domains. The following remark will elucidate this situation.

Suppose that $\mathcal{B}_1 = \langle B_1, \wedge_1, \prime_1 \rangle$ and $\mathcal{B}_2 = \langle B_2, \wedge_2, \prime_2 \rangle$ are Boolean algebras and that $\leq_1$ and $\leq_2$ are the partial orderings determined by the Boolean algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. Suppose further that $\langle \langle B_1, \leq_1 \rangle, \langle B_2, \leq_2 \rangle, J \rangle$ is a joining-system. From a formal point of view, it is possible that $\mathcal{B}_1$ and $\mathcal{B}_2$ are independent of each other, so that, for example the zero and unit elements in $\mathcal{B}_1$ are different from the zero and unit elements in $\mathcal{B}_2$.

In many applications, however, $\mathcal{B}_1$ and $\mathcal{B}_2$ are subalgebras of a common Boolean algebra $\mathcal{B} = \langle B, \wedge, ' \rangle$, and if $\perp$ is the zero element in $\mathcal{B}$ and $\top$ is the unit element in $\mathcal{B}$, then this holds in $\mathcal{B}_1$ and $\mathcal{B}_2$ as well, and, hence, $\perp$ and $\top$ are elements in the intersection of B_1 and B_2. In this case it is also natural to denote $\wedge_1$ and $\wedge_2$ with $\wedge$ and, furthermore, $\prime_1$ and $\prime_2$ with $'$. In this chapter, when there is no risk of misunderstanding, we often use $\wedge$ and $'$ (without subscript) in various Boolean algebras even when the domains and operations are different.

3.2.3 Joinings as correspondences

For a joining-system $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ (where $\mathcal{A}_1 = \langle A_1, R_1\rangle$ and $\mathcal{A}_2 = \langle A_2, R_2\rangle$), the difference in kind between relations R_1, R_2 on one hand, and J on the other, becomes more perspicuous when we introduce the distinction between ordering relations and correspondences. Obviously, both relations R_1, R_2 and the relation J are sets of ordered pairs, i.e., relations in the sense of set theory. However, while the point of each of R_1 and R_2 is to order objects in a set, the point of J is to assign objects in one set A_2 to objects in another set A_1, or vice versa.[16] This idea of J as a correspondence between sets will prove to be useful in what follows. In particular, under some general conditions, by transition through equivalence classes, an "ordering preserving" correspondence will result in an isomorphism.

The triple $\langle X, Y, \gamma\rangle$ is a *correspondence* with X as domain and Y as codomain if X and Y are sets, γ is a binary relation, and $\gamma \subseteq X \times Y$.[17] Suppose that $\langle X, Y, \gamma\rangle$ is a correspondence. If $Z \subseteq X$ we define:

$$\gamma\left[Z\right] = \{y \in Y \mid \exists x \in Z : x\gamma y\}\,.$$

If $W \subseteq Y$ then

$$\gamma^{-1}\left[W\right] = \left\{x \in X \mid \exists y \in W : y\gamma^{-1}x\right\} = \{x \in X \mid \exists y \in W : x\gamma y\}\,.$$

The correspondence $\langle X, Y, \gamma\rangle$ is on X if $\gamma^{-1}\left[Y\right] = X$, onto Y if $\gamma\left[X\right] = Y$. If there is no risk of ambiguity, we denote $\gamma\left[\{a\}\right]$ with $\gamma\left[a\right]$ and $\gamma^{-1}\left[\{b\}\right]$ with $\gamma^{-1}\left[b\right]$.

If $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ is a *Js* then $\langle A_1, A_2, J\rangle$ is a correspondence with A_1 as domain and A_2 as codomain, and we can also say that J is a correspondence from A_1 to A_2.

Definition 3.12 *Suppose that $\langle A_1, A_2, \gamma\rangle$ is a correspondence from A_1 to A_2. If $\mathcal{A}_1 = \langle A_1, R_1\rangle$ and $\mathcal{A}_2 = \langle A_2, R_2\rangle$ are quasi-orderings, we say that $\Gamma = \langle \mathcal{A}_1, \mathcal{A}_2, \gamma\rangle$ is a* quasi-ordering correspondence*, abbreviated* qo-corr.

[16] Obviously, the idea of J as a correspondence should be distinguished from the fact that there are ordering relations over the set J of ordered pairs. As we have seen, in TJS the relation of narrowness is an ordering relation over the ordered pairs in J. Another ordering relation over J (to be introduced later on) is the relation "at least as low as".

[17] If the triple $\langle X, Y, \gamma\rangle$ is a correspondence, it is sometimes more convenient to say that γ is a correspondence from X to Y and that γ^{-1} is a correspondence from Y to X. If γ is a correspondence from X to Y, Y is often called the image of X by γ, or, shorter, the γ-image of X.

If $\langle \mathcal{A}_1, \mathcal{A}_2, J \rangle$ is a *Js*, then $\langle A_1, A_2, J \rangle$ is a *qo-corr* and $J[A_1] \subseteq A_2$, where $J[A_1]$ contains the second components (belonging to A_2) of the ordered pairs that are joinings from $\mathcal{A}_1$ to $\mathcal{A}_2$. Conversely, $J^{-1}[A_2] \subseteq A_1$, where $J^{-1}[A_2]$ contains the first components (belonging to A_1) of the joinings from $\mathcal{A}_1$ to $\mathcal{A}_{2.}$. Then $J^{-1}[A_2]$ is the set of grounds and $J[A_1]$ the set of consequences of the joinings in $\langle \mathcal{A}_1, \mathcal{A}_2, J \rangle$.

The relative product of two correspondences γ and δ is denoted $\gamma|\delta$. If $\langle \mathcal{A}_1, \mathcal{A}_2, J \rangle$ is a joining-system, then $R_1|J|R_2 = J$ and, therefore, J can be said to "absorb" R_1 and R_2. Note that $x_1(R_1|J|R_2)x_2$ iff $\exists y_1, y_2 : x_1 R_1 y_1 \ \& \ y_1 J y_2 \ \& \ y_2 R_2 x_2$.

3.2.4 Order-preservation and order-similarity

The notion of *qo-corr* is a basis for the notions of "order-preservation" and "order-similarity". Suppose $\mathcal{A}_1 = \langle A_1, R_1 \rangle$ and $\mathcal{A}_2 = \langle A_2, R_2 \rangle$ are two strata, and that J is a *qo-corr* from $\mathcal{A}_1$ to $\mathcal{A}_2$. If $\langle \mathcal{A}_1, \mathcal{A}_2, J \rangle$ is order-preserving, Q_1-similar grounds in A_1 have the same consequences in A_2, Q_2-similar consequences in A_2 have the same grounds in A_1, and if $\langle a_1, a_2 \rangle$, $\langle b_1, b_2 \rangle$ are joinings from $\mathcal{A}_1$ to $\mathcal{A}_2$, then the R_1-structure on $\{a_1, b_1\}$ is similar to the R_2-structure on $\{a_2, b_2\}$ insofar as $a_1 R_1 b_1$ iff $a_2 R_2 b_2$. The general definition is as follows.

Definition 3.13 *Suppose that* $\Gamma = \langle \langle A_1, R_1 \rangle, \langle A_2, R_2 \rangle, \gamma \rangle$ *is a* qo-corr. *We say that* Γ *is* order-preserving *if the following holds for* $a_1, b_1 \in A_1$ *and* $a_2, b_2 \in A_2$*:*

(1) *If* $a_1 Q_1 b_1$ *then* ($a_1 \gamma a_2$ *iff* $b_1 \gamma a_2$).

(2) *If* $a_2 Q_2 b_2$ *then* ($a_1 \gamma a_2$ *iff* $a_1 \gamma b_2$).

(3) *If* $a_1 \gamma a_2$ *and* $b_1 \gamma b_2$ *then* $a_1 R_1 b_1$ *iff* $a_2 R_2 b_2$.

Definition 3.14 *Two quasi-orderings* $\langle A_1, R_1 \rangle$ *and* $\langle A_2, R_2 \rangle$ *are said to be* order-similar *if there is* $\gamma \subseteq A_1 \times A_2$ *such that* $\langle \langle A_1, R_1 \rangle, \langle A_2, R_2 \rangle, \gamma \rangle$ *is an order-preserving* qo-corr *on* A_1 *onto* A_2.

The notion of "order-preserving *qo-corr*" is elucidated by the fact that by transition from quasi-orderings to equivalence classes you get an isomorphism between the resulting structures; also, if there is an isomorphism between the equivalence classes, there is order-preservation between the quasi-orderings.

Theorem 3.15 *Suppose that* $\langle \langle A_1, R_1 \rangle, \langle A_2, R_2 \rangle, \gamma \rangle$ *is a* qo-corr *on* A_1 *onto* A_2. *Let* $[a]_i$ $[b]_i$ *be the equivalence-classes with respect to* Q_i *generated by* a *and* b, *respectively* ($i = 1, 2$). *Let further* $A_1^* = \left\{ [a]_1 \mid a \in \gamma^{-1}[A_2] \right\}$

and $A_2^* = \{[a]_2 \mid a \in \gamma[A_1]\}$ *and let* R_i^* *be defined as follows:* $[a]_i R_i^* [b]_i$ *iff* aR_ib.

(1) *Suppose that* $\langle\langle A_1, R_1\rangle, \langle A_2, R_2\rangle, \gamma\rangle$ *is an order-preserving* qo-corr *and let* γ^* *be defined by* $[a_1]_1 \gamma^* [a_2]_2$ *iff* $a_1\gamma a_2$. *Then* γ^* *is an isomorphism on* $\langle A_1^*, R_1^*\rangle$ *onto* $\langle A_2^*, R_2^*\rangle$. *If* $\langle A_1, R_1\rangle$ *and* $\langle A_2, R_2\rangle$ *are quasi-lattices (see Definition 3.4), then* γ^* *is an isomorphism on the lattice* $\langle A_1^*, R_1^*\rangle$ *onto the lattice* $\langle A_2^*, R_2^*\rangle$.

(2) *If* φ *is an isomorphism on* $\langle A_1^*, R_1^*\rangle$ *onto* $\langle A_2^*, R_2^*\rangle$, *then*

$$\langle\langle A_1, R_1\rangle, \langle A_2, R_2\rangle, \gamma\rangle$$

is an order-preserving qo-corr *on* A_1 *onto* A_2, *where* γ *is defined by* $a_1\gamma a_2$ *iff* $\varphi([a_1]_1) = [a_2]_2$.

3.2.5 Joining-closure and the generating of joining-spaces

An important aspect of TJS is that it gives a method (the forming of a "joining-closure") for representing an "elaborated" version of a set of "crude" conditional norms. Suppose that $\mathcal{A}_1$ is a quasi-ordering of grounds and $\mathcal{A}_2$ is a quasi-ordering of consequences. Let us suppose that K is a set of conditional norms with the antecedents taken from A_1 and the consequences taken from A_2. Hence, $K \subseteq A_1 \times A_2$ and K is a correspondence from A_1 to A_2. The set K can be thought of as a crude representation of a normative system $\mathcal{N}$. Then we can generate a set K^* by forming the "joining closure" of K such that $\langle\mathcal{A}_1, \mathcal{A}_2, K^*\rangle$ is a joining-system, which will be explained below.

The next theorem shows that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are quasi-orderings and

$$\mathcal{J} = \{J \subseteq A_1 \times A_2 \mid \langle\mathcal{A}_1, \mathcal{A}_2, J\rangle \text{ is a } Js\},$$

then $\mathcal{J}$ is a closure system.[18] Note that $\mathcal{J}$ is the family of all joining-spaces from $\mathcal{A}_1$ to $\mathcal{A}_2$.

Theorem 3.16 *If* $\mathcal{J} = \{J \subseteq A_1 \times A_2 \mid \langle\mathcal{A}_1, \mathcal{A}_2, J\rangle$ *is a Js*$\}$ *and* $\mathcal{K} \subseteq \mathcal{J}$, *then* $\cap\mathcal{K} \in \mathcal{J}$.

Proof. If $\cap\mathcal{K} = \varnothing$, then $\langle\mathcal{A}_1, \mathcal{A}_2, \cap\mathcal{K}\rangle$ is the empty joining-system and hence $\cap\mathcal{K} \in \mathcal{J}$. Now suppose that $\cap\mathcal{K} \neq \varnothing$.

(I) Firstly, we prove that condition (1) in the definition of a joining-system is satisfied. Suppose therefore that $b_i, c_i \in A_i$ for $i = 1, 2$ and $\langle b_1, b_2\rangle \in \cap\mathcal{K}$ and $\langle b_1, b_2\rangle \trianglelefteq \langle c_1, c_2\rangle$. Let $K \in \mathcal{K}$. Then $\cap\mathcal{K} \subseteq K$ and thus $\langle b_1, b_2\rangle \in K$.

[18] For definition and results of closure systems, see for example [Grätzer, 1979, p. 23f.].

Since $K \in \mathcal{J}$ and $\langle b_1, b_2 \rangle \trianglelefteq \langle c_1, c_2 \rangle$ it follows that $\langle c_1, c_2 \rangle \in K$. Hence, for all $K \in \mathcal{K}$, $\langle c_1, c_2 \rangle \in K$ which implies $\langle c_1, c_2 \rangle \in \cap\mathcal{K}$.

(II) Secondly, we prove that condition (2) in the definition of a joining-system is satisfied. Suppose that $C_1 \subseteq A_1$, $b_2 \in A_2$, and $\langle c_1, b_2 \rangle \in \cap\mathcal{K}$ for all $c_1 \in C_1$. Then $\langle c_1, b_2 \rangle \in K$ for all $c_1 \in C_1$ and $K \in \mathcal{K}$. Since $K \in \mathcal{J}$ it follows that $\langle a_1, b_2 \rangle \in K$ for all $a_1 \in \operatorname{lub}_{R_1} C_1$. Hence, for all $K \in \mathcal{K}$, $\langle a_1, b_2 \rangle \in K$ for all $a_1 \in \operatorname{lub}_{R_1} C_1$, which implies $\langle a_1, b_2 \rangle \in \cap\mathcal{K}$ for all $a_1 \in \operatorname{lub}_{R_1} C_1$.

(III) Thirdly, we prove that condition (3) in the definition of a joining-system is satisfied. Suppose that $C_2 \subseteq A_2$, $b_1 \in A_1$, and $\langle b_1, c_2 \rangle \in \cap\mathcal{K}$ for all $c_2 \in C_2$. Then $\langle b_1, c_2 \rangle \in K$ for all $c_2 \in C_2$ and $K \in \mathcal{K}$. Since $K \in \mathcal{J}$ it follows that $\langle b_1, a_2 \rangle \in K$ for all $a_2 \in \operatorname{glb}_{R_2} C_2$. Hence, for all $K \in \mathcal{K}$, $\langle b_1, a_2 \rangle \in K$ for all $a_2 \in \operatorname{glb}_{R_2} C_2$, which implies $\langle b_1, a_2 \rangle \in \cap\mathcal{K}$ for all $a_2 \in \operatorname{glb}_{R_2} C_2$. ■

From the theorem follows that if $K \subseteq A_1 \times A_2$ and

$$[K]_{\mathcal{J}} = \cap \{J \mid J \in \mathcal{J},\ J \supseteq K\},$$

then $[K]_{\mathcal{J}}$ is the joining-space, here called the *joining-closure*, over $\mathcal{A}_1$ and $\mathcal{A}_2$ *generated* by K. (Note that since $A_1 \times A_2$ is a joining space, $\{J \mid J \in \mathcal{J},\ J \supseteq K\} \neq \varnothing$.)

If J is the joining-closure from $\mathcal{A}_1$ to $\mathcal{A}_2$ generated by K but J is not generated by any proper subset of K, then we say that J is the joining-closure *non-redundantly generated* by K.

3.3 Weakest grounds, strongest consequences and minimal joinings

3.3.1 Weakest grounds and strongest consequences

Definition 3.17 *Suppose that $\mathcal{S} = \langle \mathcal{A}_1, \mathcal{A}_2, J \rangle$ is a joining-system, and that $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$. Then,*

1. *$a_1 \in C_1 \subseteq A_1$ is one of the* weakest grounds *of $a_2 \in A_2$ in C_1 with respect to $\mathcal{S}$, which is denoted* $\mathrm{WG}_{\mathcal{S}}(a_1, a_2, C_1)$*, if*

$$\langle a_1, a_2 \rangle \in J \text{ and, for any } b_1 \in C_1,$$
$$\text{it holds that } \langle b_1, a_2 \rangle \in J \text{ implies } b_1 R_1 a_1.$$

2. *$a_2 \in C_2 \subseteq A_2$ is one of the* strongest consequences *of $a_1 \in A_1$ in C_2 with respect to $\mathcal{S}$, which is denoted* $\mathrm{SC}_{\mathcal{S}}(a_2, a_1, C_2)$*, if*

$$\langle a_1, a_2 \rangle \in J, \text{ and, for any } b_2 \in C_2,$$
$$\text{it holds that } \langle a_1, b_2 \rangle \in J \text{ implies } a_2 R_2 b_2.$$

In Section 3.3.2, the interrelationship between minimal joinings and weakest grounds, strongest consequences will be further developed. Below, however, are some basic results. (Cf. [Lindahl and Odelstad, 2011, sect. 3.2].)

Theorem 3.18 *Let* $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ *be a joining-system.*
(1) *Suppose that* $\mathrm{WG}\,(a_1, a_2, A_1)$ *and* $\mathrm{WG}\,(b_1, b_2, A_1)$*. If* $a_2R_2b_2$*, then* $a_1R_1b_1$*.*
(2) *Suppose that* $\mathrm{SC}\,(a_2, a_1, A_2)$ *and* $\mathrm{SC}\,(b_2, b_1, A_2)$ *If* $a_1R_1b_1$*, then* $a_2R_2b_2$*.*
(3) *Suppose that* $\mathrm{WG}\,(a_1, a_2, A_1)$ *and* $\mathrm{WG}\,(b_1, b_2, A_1)$*. For all* $c_1 \in A_1$ *and* $c_2 \in A_2$*, if* $c_1 \in \mathrm{glb}_{R_1}\{a_1, b_1\}$ *and* $c_2 \in \mathrm{glb}_{R_2}\{a_2, b_2\}$*, then* $\mathrm{WG}\,(c_1, c_2, A_1)$*.*
(4) *Suppose that* $\mathrm{SC}\,(a_2, a_1, A_2)$ *and* $\mathrm{SC}\,(b_2, b_1, A_2)$*. For all* $c_1 \in A_1$ *and* $c_2 \in A_2$*, if* $c_1 \in \mathrm{lub}_{R_1}\{a_1, b_1\}$ *and* $c_2 \in \mathrm{lub}_{R_2}\{a_2, b_2\}$*, then* $\mathrm{SC}\,(c_2, c_1, A_2)$*.*

Proof. We prove (3). Note that a_1Ja_2 and b_1Jb_2. Suppose that $c_1 \in \mathrm{glb}_{R_1}\{a_1, b_1\}$ and $c_2 \in \mathrm{glb}_{R_2}\{a_2, b_2\}$. Hence, c_1Ja_2 and c_1Jb_2 and according to condition (3) in the definition of a joining-system, c_1Jc_2. Suppose that d_1Jc_2. Then d_1Ja_2 and d_1Jb_2, and since $\mathrm{WG}\,(a_1, a_2, A_1)$ and $\mathrm{WG}\,(b_1, b_2, A_1)$ it follows that $d_1R_1a_1$ and $d_1R_1b_1$ which implies that $d_1R_1c_1$. Thus $\mathrm{WG}\,(c_1, c_2, A_1)$. ■

Item (1) in Theorem 3.18 is illustrated by Figure 5, and item (2) by Figure 6.

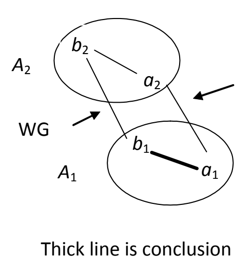


Figure 5

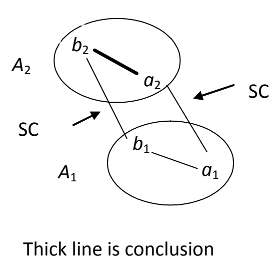


Figure 6

Theorem 3.19 *Let* $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ *be a joining-system.*
(1) *Suppose that* $\mathcal{A}_1$ *is a complete quasi-lattice (see Definition 3.4). Then* $\mathrm{WG}\,(a_1, a_2, A_1)$ *iff* $a_1 \in \mathrm{lub}_{R_1} J^{-1}[a_2]$*.*
(2) *Suppose that* $\mathcal{A}_2$ *is a complete quasi-lattice. Then* $\mathrm{SC}\,(a_2, a_1, A_2)$ *iff* $a_2 \in \mathrm{glb}_{R_2} J[a_1]$*.*

Proof. We prove (1) above. (i) Suppose that $\mathrm{WG}\,(a_1, a_2, A_1)$. Hence, $a_1 \in J^{-1}[a_2]$. Since $\mathcal{A}_1$ is a complete quasi-lattice it follows that there is $b_1 \in \mathrm{lub}_{R_1} J^{-1}[a_2]$ and $a_1 R_1 b_1$. From condition (2) of a joining-system it follows that $\langle b_1, a_2\rangle \in J$. Since $\mathrm{WG}(a_1, a_2, A_1)$, it follows that $b_1 R_1 a_1$. Together with $a_1 R_1 b_1$, this implies $a_1 Q_1 b_1$. Thus $a_1 \in \mathrm{lub}_{R_1} J^{-1}[a_2]$. (ii) Suppose that $a_1 \in \mathrm{lub}_{R_1} J^{-1}[a_2]$. If $\langle b_1, a_2\rangle \in J$ then $b_1 \in J^{-1}[a_2]$ and hence $b_1 R_1 a_1$. From this follows that $\mathrm{WG}\,(a_1, a_2, A_1)$. (Note that this part of the proof does not require that $\mathcal{A}_2$ is a complete quasi-lattice.) The proof of (2) is analogous. ∎

3.3.2 Minimal joinings

Minimal joinings in a *Js* will be a central theme in the subsequent presentation. The formal definition is as follows (we recall the definition of "minimal element" with respect to narrowness in Definition 3.9).

Definition 3.20 *Suppose that* $\langle \mathcal{A}_1, \mathcal{A}_2, K\rangle$ *is a* qo-corr*. A* minimal element *in* $\langle \mathcal{A}_1, \mathcal{A}_2, K\rangle$ *is a minimal element* $\langle a_1, a_2\rangle$ *in* K *with respect to* $\mathcal{A}_1$ *and* $\mathcal{A}_2$*. The set of minimal elements in* $\langle \mathcal{A}_1, \mathcal{A}_2, K\rangle$ *is denoted* $\min\langle \mathcal{A}_1, \mathcal{A}_2, K\rangle$ *or just* $\min K$*.*

If $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ is a joining-system, then the elements in $\min J$ are often called minimal joinings. The connection between the notion of minimal joining on one hand and the notions of weakest ground and strongest consequence on the other side is made clear in the following theorem.

Theorem 3.21 *Suppose that* $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ *is a joining-system. Then* $\langle a_1, a_2\rangle \in \min J$ *iff* $\mathrm{WG}\,(a_1, a_2, A_1)$ *and* $\mathrm{SC}\,(a_2, a_1, A_2)$*. See Figure 7.*

A proof of the theorem under the assumption that $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ is a Boolean joining-system is given in [Lindahl and Odelstad, 2011, theorem 36, p. 126], but it is easy to see that the theorem holds even if $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ is a mere joining-system.

3.4 Connectivity

As stated in the introductory Section 2.1.2, if a normative system fulfils a requirement called "connectivity", any norm in the system will always be implied by a minimal joining. Therefore, the idea of connectivity will be essential in the theory of minimal joinings to be developed in the next subsections. The definition of connectivity is given next.

Definition 3.22 *A* qo-corr $\langle \mathcal{A}_1, \mathcal{A}_2, K\rangle$ *such that* K *is an up-set with respect to* $\trianglelefteq$ *satisfies* connectivity *if whenever* $\langle c_1, c_2\rangle \in K$ *there is* $\langle b_1, b_2\rangle \in K$ *such that* $\langle b_1, b_2\rangle$ *is a minimal element in* K *with respect to* $\trianglelefteq$ *and* $\langle b_1, b_2\rangle \trianglelefteq \langle c_1, c_2\rangle$*.*

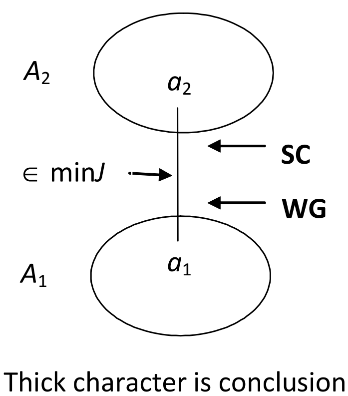


Figure 7

Definition 3.23 *Suppose that* $\langle \mathcal{A}_1, \mathcal{A}_2, K\rangle$ *is a* qo-corr*. Then the set*

$$\{\langle a_1, a_2\rangle \in A_1 \times A_2 \mid \exists \langle b_1, b_2\rangle \in K : \langle b_1, b_2\rangle \trianglelefteq \langle a_1, a_2\rangle\}$$

is called the enclosure of K *and is denoted* $\uparrow K$*.*

Note that $\uparrow K$ is an up-set (with respect to $\trianglelefteq$) and the smallest up-set containing K. (For the notion of up-set see Definition 3.10 in Section 3.2.1.) To use an expression from lattice theory, $\uparrow K$ is read 'up K' (with respect to $\trianglelefteq$). (See [Davey and Priestley, 2002, p. 20].) Note also that K is an up-set if and only if $K = \uparrow K$.

Theorem 3.24 *Suppose that* $\langle \mathcal{A}_1, \mathcal{A}_2, K\rangle$ *is a* qo-corr *such that* K *is an up-set with respect to* $\trianglelefteq$*. Then* $\langle \mathcal{A}_1, \mathcal{A}_2, K\rangle$ *satisfies connectivity iff* $K = \uparrow \min K$*.*

Proof. (I) Suppose $\langle \mathcal{A}_1, \mathcal{A}_2, K\rangle$ satisfies connectivity. (i) Suppose $\langle a_1, a_2\rangle \in K$. Then there is $\langle b_1, b_2\rangle \in \min K$ such that $\langle b_1, b_2\rangle \trianglelefteq \langle a_1, a_2\rangle$ and hence $\langle a_1, a_2\rangle \in\uparrow \min K$. This shows that $K \subseteq\uparrow \min K$. (ii) Suppose $\langle a_1, a_2\rangle \in\uparrow \min K$. Then there is $\langle b_1, b_2\rangle \in \min K$ such that $\langle b_1, b_2\rangle \trianglelefteq \langle a_1, a_2\rangle$. Since $\langle \mathcal{A}_1, \mathcal{A}_2, K\rangle$ is a *qo-corr* such that K is an up-set with respect to $\trianglelefteq$, $\langle a_1, a_2\rangle \in K$. Hence, $\uparrow \min K \subseteq K$.

(II) Suppose that $K =\uparrow \min K$ and that $\langle a_1, a_2\rangle \in K$. Then $\langle a_1, a_2\rangle \in\uparrow \min K$ and there is $\langle b_1, b_2\rangle \in \min K$ such that $\langle b_1, b_2\rangle \trianglelefteq \langle a_1, a_2\rangle$. This shows that $\langle \mathcal{A}_1, \mathcal{A}_2, K\rangle$ satisfies connectivity. ■

If a joining-system satisfies connectivity, then the set of minimal joinings determines the system in an interesting way, which will be explained below.

Corollary 3.25 *If the joining–system* $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ *satisfies connectivity, then* $J = \uparrow \min J$, *that is,*

$$J = \{\langle a_1, a_2\rangle \in A_1 \times A_2 \mid \exists \langle b_1, b_2\rangle \in \min J : \langle b_1, b_2\rangle \trianglelefteq \langle a_1, a_2\rangle\}.$$

The corollary shows that there is an interesting way of representing a normative system in terms of $\trianglelefteq$ -minimal elements. This way of representing is different from the method of "joining-closure" presented above in Section 3.2.5 and we will here develop it a little further.

Note that we have not so far said anything about how to get a joining-system using the enclosure of a *qo-corr* (Definition 3.23). We will return to this problem in Section 3.6.

Theorem 3.26 *If* $\mathcal{A}_1 = \langle A_1, R_1\rangle$ *and* $\mathcal{A}_2 = \langle A_2, R_2\rangle$ *are complete quasi-lattices (see Definition 3.4, Section 3.1.2), and* $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ *is a joining-system, then* $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ *satisfies connectivity.*

Proof. Suppose $\langle c_1, c_2\rangle \in J$. Let $X_1 = \{x_1 \in A_1 | \langle x_1, c_2\rangle \in J\}$. Since $\mathcal{A}_1$ is a complete quasi-lattice it holds that $\mathrm{lub}\, X_1 \neq \varnothing$. Let $b_1 \in \mathrm{lub}\, X_1$. From (2) in the definition of a joining-system follows that $\langle b_1, c_2\rangle \in J$ and hence $b_1 \in X_1$. Let $X_2 = \{x_2 \in A_2 | \langle b_1, x_2\rangle \in J\}$. Since $\langle b_1, c_2\rangle \in J$, $X_2 \neq \varnothing$. $\mathcal{A}_2$ is a complete quasi-lattice and therefore it holds that glb $X_2 \neq \varnothing$. Let $b_2 \in$ glb X_2. From (3) in the definition of a joining-system follows that $\langle b_1, b_2\rangle \in J$ and hence $b_2 \in X_2$. Since $c_1 \in X_1$ and $b_1 \in \mathrm{lub}\, X_1$ then $c_1 R_1 b_1$. And since $c_2 \in X_2$ and $b_2 \in \mathrm{glb}\, X_2$ then $b_2 R_2 c_2$. Hence, $\langle b_1, b_2\rangle \trianglelefteq \langle c_1, c_2\rangle$.

Suppose now that $\langle a_1, a_2\rangle \in J$ and $\langle a_1, a_2\rangle \trianglelefteq \langle b_1, b_2\rangle$. Thus $c_1 R_1 b_1 R_1 a_1$ and $a_2 R_2 b_2 R_2 c_2$, which implies that $\langle a_1, a_2\rangle \trianglelefteq \langle a_1, c_2\rangle$ and $\langle a_1, a_2\rangle \trianglelefteq \langle b_1, a_2\rangle$. According to condition (1) in the definition of a joining-system, it follows that $\langle a_1, c_2\rangle, \langle b_1, a_2\rangle \in J$ and thus $a_1 \in X_1$ and $a_2 \in X_2$. Since $b_1 \in \mathrm{ub}_{R_1} X_1$ it follows that $a_1 R_1 b_1$, and since $b_2 \in \mathrm{lb}_{R_2} X_2$ it follows that $b_2 R_2 a_2$. Hence, $a_1 Q_1 b_1$ and $a_2 Q_2 b_2$, and we conclude that $\langle b_1, b_2\rangle$ is a minimal element in $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$. ■

The next theorem states that if connectivity holds, then a weakest ground of an element is the bottom of a minimal joining and a strongest consequence of an element is the top of a minimal joining.

Theorem 3.27 *Suppose that* $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ *is a joining-system which satisfies connectivity (see Definition 3.22). Then:*

1. If $\mathrm{WG}(a_1, a_2, A_1)$ then there is $b_2 \in A_2$ such that $\langle a_1, b_2 \rangle \in \min J$ and $b_2 R_2 a_2$.
2. If $\mathrm{SC}(a_2, a_1, A_2)$ then there is $b_1 \in A_1$ such that $\langle b_1, a_2 \rangle \in \min J$ and $a_1 R_1 b_1$.

(For a proof, see [Odelstad, 2008, pp. 50f.].)

Considering a joining-system $\langle \mathcal{A}_1, \mathcal{A}_2, J \rangle$, a useful device is the introduction of projections $\pi_1[J] \subseteq A_1$ and $\pi_2[J] \subseteq A_2$, which implies that each $a_1 \in \pi_1[J]$ is a "ground" for some element a_2 of A_2 and, conversely, each $a_2 \in \pi_2[J]$ is a "consequence" of some element a_1 of A_1. The general definition is as follows.

Definition 3.28 *For sets A_1 and A_2, if $X \subseteq A_1 \times A_2$ then for $i = 1, 2$, $\pi_i : X \to A_i$ is such that $\pi_i(x_1, x_2) = x_i$ is the* projection *of X on the ith coordinate.*

Note that if $X \subseteq A_1 \times A_2$ then $\pi_1[X] = \{x_1 \in A_1 \mid \exists x_2 \in A_2 : \langle x_1, x_2 \rangle \in X\}$

$$\pi_2[X] = \{x_2 \in A_2 \mid \exists x_1 \in A_1 : \langle x_1, x_2 \rangle \in X\}$$

The subsequent Theorem 3.30 might be easier to grasp if we first consider the special case of a joining-system $\langle \mathcal{L}_1, \mathcal{L}_2, J \rangle$ where $\mathcal{L}_1 = \langle L_1, \wedge, \vee \rangle$, $\mathcal{L}_2 = \langle L_2, \wedge, \vee \rangle$ are lattices and $\leq_1$, $\leq_2$ are the partial orderings determined by these lattices. Then, according to Theorem 3.30, if $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in \min J$, there is $c_2 \in L_2$, $d_1 \in L_1$ such that

(1) $\langle a_1 \wedge b_1, c_2 \rangle \in \min J$,

(2) $\langle d_1, a_2 \vee b_2 \rangle \in \min J$,

(3) $c_2 \leq_2 a_2 \wedge b_2$,

(4) $a_1 \vee b_1 \leq_1 d_1$.

The following theorem is used in the proof of Theorem 3.30.

Theorem 3.29 *Suppose that $\langle \mathcal{A}_1, \mathcal{A}_2, J \rangle$ is a joining-system that satisfies connectivity. Then the following holds:*

(i) *If $\langle a_1, a_2 \rangle \in \min J$, then $\langle a_1, b_2 \rangle \in J$ implies $a_2 R_2 b_2$ and $\langle b_1, a_2 \rangle \in J$ implies $b_1 R_1 a_1$. (See Figure 8 on page 581.)*

(ii) *If $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in \min J$ then $a_1 R_1 b_1$ iff $a_2 R_2 b_2$.*

(iii) *If $\langle a_1, a_2 \rangle \in \min J$ then $\langle a_1, b_2 \rangle \in \min J$ implies $a_2 Q_2 b_2$ and $\langle b_1, a_2 \rangle \in \min J$ implies $a_1 Q_1 b_1$. (See Figure 9 on page 582.)*

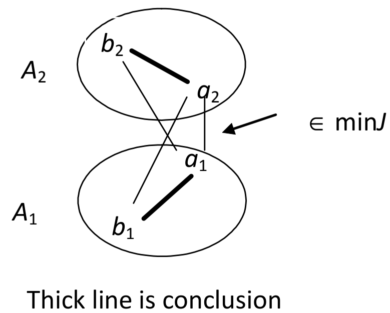


Figure 8

(For a proof, see [Odelstad, 2008, p. 51].)

Theorem 3.30 *Suppose that* $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ *is a joining-system and that* $\mathcal{A}_1 = \langle A_1, R_1\rangle$ *and* $\mathcal{A}_2 = \langle A_2, R_2\rangle$ *are complete quasi-lattices. If* $X \subseteq \min J$ *and* $X \neq \varnothing$ *then the following holds:*

(1) There is $c_2 \in A_2$ *such that for all* $a_1 \in \mathrm{glb}_{R_1} \pi_1 [X]$, $\langle a_1, c_2\rangle \in \min J$, *and, furthermore, it holds that* $c_2 R_2 a_2$ *for all* $a_2 \in \mathrm{glb}_{R_2} \pi_2 [X]$.

(2) There is $d_1 \in A_1$ *such that for all* $b_2 \in \mathrm{lub}_{R_2} \pi_2 [X]$, $\langle d_1, b_2\rangle \in \min J$, *and, furthermore, it holds that* $b_1 R_1 d_1$ *for all* $b_1 \in \mathrm{lub}_{R_1} \pi_1 [X]$.

Proof. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are complete quasi-lattices it follows from Theorem 3.26 that $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ satisfies connectivity.
(I) We prove (1). Since $\mathcal{A}_1$ is a complete quasi-lattice, it follows that there is $a_1 \in \mathrm{glb}_{R_1} \pi_1 [X]$. Suppose that $x_2 \in \pi_2 [X]$. Then there is $x_1 \in \pi_1 [X]$ such that $\langle x_1, x_2\rangle \in X$ and $\langle x_1, x_2\rangle \trianglelefteq \langle a_1, x_2\rangle$. Since $X \subseteq J$ it follows that $\langle a_1, x_2\rangle \in J$ and this holds for all $x_2 \in \pi_2 [X]$. Since $\mathcal{A}_2$ is a complete quasi-lattice, it follows that $\mathrm{glb}_{R_2} \pi_2 [X] \neq \varnothing$. Let $a_2 \in \mathrm{glb}_{R_2} \pi_2 [X]$. From condition (3) in the definition of a *Js* it follows that $\langle a_1, a_2\rangle \in J$. Since J satisfies connectivity it follows that there is $\langle c_1, c_2\rangle \in \min J$ such that $\langle c_1, c_2\rangle \trianglelefteq \langle a_1, a_2\rangle$. Let $\langle z_1, z_2\rangle \in X$, which implies that $\langle z_1, z_2\rangle \in \min J$ and since $z_2 \in \pi_2 [X]$ and $a_2 \in \mathrm{glb}_{R_2} \pi_2 [X]$ it follows that $a_2 R_2 z_2$. Furthermore, $c_2 R_2 a_2$ and thus $c_2 R_2 z_2$, which implies according to (ii) in theorem 3.29, that $c_1 R_1 z_1$. Hence, $c_1 \in \mathrm{lb}_{R_1} \pi_1 [X]$. Since $a_1 \in \mathrm{glb}_{R_1} \pi_1 [X]$ it follows that $c_1 R_1 a_1$, and since $a_1 R_1 c_1$ this implies $a_1 Q_1 c_1$. This shows that $\langle a_1, c_2\rangle \in \min J$. Note that $c_2 R_2 a_2$.
(II) The proof of (2) is analogous. ■

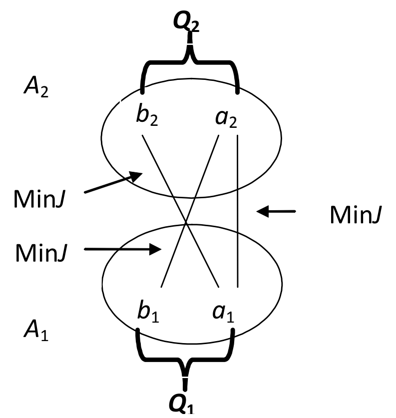


Figure 9

An illustration in a lattice framework of (1) and (2) in Theorem 3.30 is provided in Figures 10 on page 583 and Figure 11 on page 584, respectively.

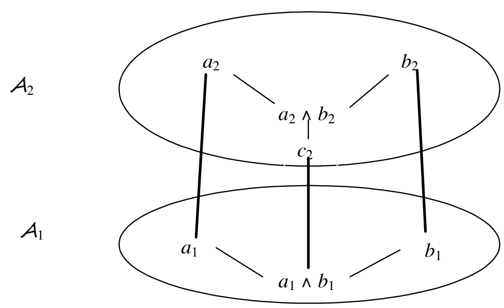
3.5 Lowerness

In the literature on partial orderings, the notion "coordinatewise ordering" of a Cartesian product of partial ordered sets is introduced (see for example [Davey and Priestley, 2002, p. 18].) It is straight forward to generalize this notion to quasi-ordered sets. This is done in the definition below. With the interpretation of TJS in this chapter as a theory of normative systems, we call the relation "coordinatewise ordering" the *lowerness-relation.*

Definition 3.31 *The* lowerness relation determined by *the quasi-orderings* $\langle A_1, R_1\rangle$ *and* $\langle A_2, R_2\rangle$ *is the binary relation* $\precsim$ *on* $A_1 \times A_2$ *such that for all* $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in A_1 \times A_2$

$$\langle a_1, a_2\rangle \precsim \langle b_1, b_2\rangle \quad \textit{iff} \quad a_1 R_1 b_1 \quad \textit{and} \quad a_2 R_2 b_2.$$

For elements in $A_1 \times A_2$ we read $\precsim$ as "at least as low as". If j_1 and j_2 are elements in $A_1 \times A_2$, then j_1 is at least as low as j_2, i.e. $j_1 \precsim j_2$, if the "bottom" of j_1 is at least as low as, i.e. stands in the relation R_1 to, the "bottom" of j_2, and the "top" of j_1 is at least as low as, i.e. stands in the relation R_2 to, the "top" of j_2. See Figure 12 on page 585. (As a contrast, see Figure 4 on page 569.) Note that $\precsim$ is a quasi-ordering, i.e. transitive



$\mathrm{glb}_{R_1}\pi_1\{\langle a_1,a_2\rangle,\langle b_1,b_2\rangle\} = \mathrm{glb}_{R_1}\{a_1,b_1\}=\{a_1 \wedge b_1\}$
$\mathrm{glb}_{\leq/\mathrm{min}J}\,\{\langle a_1,a_2\rangle,\langle b_1,b_2\rangle\} = \{\langle a_1 \wedge b_1,c_2\rangle\}, \pi_1\,[\mathrm{glb}_{\leq/\mathrm{min}J}\,\{\langle a_1,a_2\rangle,\langle b_1,b_2\rangle\}] = \{a_1 \wedge b_1\}$

Figure 10

and reflexive. Let $\sim$ denote the equality part of $\precsim$ and $\prec$ the strict part of $\precsim$. Then the following holds:

$\langle a_1, a_2\rangle \sim \langle b_1, b_2\rangle$ iff $b_1Q_1a_1$ & $a_2Q_2b_2$
$\langle a_1, a_2\rangle \prec \langle b_1, b_2\rangle$ iff $(a_1P_1b_1$ & $a_2R_2b_2)$ or $(a_1R_1b_1$ & $a_2P_2b_2)$

where Q_i is the equality-part of R_i and P_i is the strict part of R_i.

The structure of the minimal joinings in a joining-system is similar to the structure of their "bottoms" and "tops". We recall the definition of projections π_i (Definition 3.28 in Section 3.4).

Theorem 3.32 *Suppose that $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ is a joining-system that satisfies connectivity (See Definition 3.22). Then for $i = 1, 2$, $\pi_i : \min J \longrightarrow \pi_i [\min J]$ is surjective, and the following holds:*

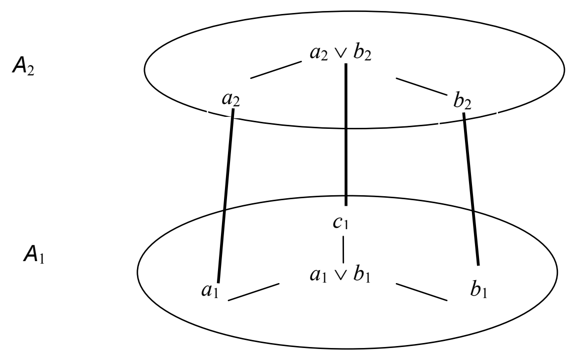
$$\text{for all } \alpha, \beta \in \min J,\ \alpha \precsim \beta \text{ iff } \pi_i(\alpha)\, R_i \pi_i(\beta)\,.$$

Proof. Follows from Theorem 3.29, (ii). ■

Corollary 3.33 *If $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ is a joining-system satisfying connectivity, then*

$$\langle\langle \pi_1 [\min J] ,R_1\rangle , \langle \pi_2 [\min J] ,R_2\rangle , \min J\rangle$$

is an order-preserving quasi-order correspondence (cf. Definitions 3.13 and 3.12).



$\mathrm{lub}_{R2}\pi_2\{<a_1,a_2>,<b_1,b_2>\} = \mathrm{lub}_{R2}\,\{a_2,b_2\} = \{a_2 \vee b_2\}$

$[\mathrm{lub}_{\leq/\min J}\ \{<a_1,a_2>,<b_1,b_2>\}] = \{<c_1,\, a_2 \vee b_2>\},\ \ \pi_2[\mathrm{lub}_{\leq/\min J}\ \{<a_1,a_2>,<b_1,b_2>\}] = \{a_2 \vee b_2\}$

Figure 11

The corollary says that in a joining-system $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$, the R_1-structure of set of "bottoms" of $\min J$ is order similar to the R_2-structure of the set of "tops" of $\min J$. (See Theorem 3.15 for how this result can be expressed in terms of the notion of isomorphism.)

3.5.1 A remark on the interrelation between narrowness and lowerness

Given the quasi-orderings $\langle A_1, R_1\rangle$ and $\langle A_2, R_2\rangle$, we have introduced two quasi-orderings on $A_1 \times A_2$, viz. the narrowness relation $\trianglelefteq$ and the lowerness relation $\precsim$. The interrelation between these two orderings is of great interest in the study of joining-systems.

How narrowness and lowerness are connected becomes more transparent if we if we restrict ourselves to consider lattices instead of quasi-orderings. Suppose that $\langle L_1, \leq_1\rangle$ and $\langle L_2, \leq_2\rangle$ are lattices. Let $\precsim$ be the lowerness-relation with respect to $\leq_1$ and $\leq_2$, i.e. for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in L_1 \times L_2$

$$\langle a_1, a_2\rangle \precsim \langle b_1, b_2\rangle \quad \text{iff} \quad a_1 \leq_1 b_1 \quad \text{and} \quad a_2 \leq_2 b_2.$$

Then $\langle L_1 \times L_2, \precsim\rangle$ is a lattice and is the product of $\langle L_1, \leq_1\rangle$ and $\langle L_2, \leq_2\rangle$. Let $\langle L_1, \wedge_1, \vee_1\rangle$ and $\langle L_2, \wedge_2, \vee_2\rangle$ be the algebraic formulation of $\langle L_1, \leq_1\rangle$ and $\langle L_2, \leq_2\rangle$ respectively. Define

$$\begin{pmatrix} \wedge_2 \\ \wedge_1 \end{pmatrix} : L_1 \times L_2 \longrightarrow L_1 \times L_2$$

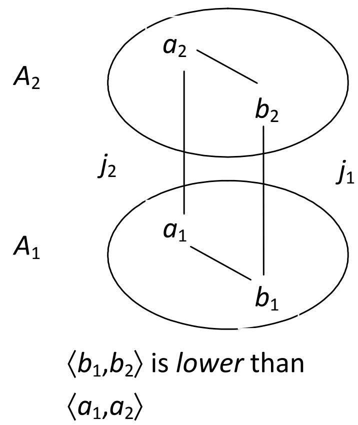


Figure 12

such that

$$\langle a_1, a_2\rangle \begin{pmatrix}\wedge_2\\ \wedge_1\end{pmatrix} \langle b_1, b_2\rangle = \langle a_1 \wedge_1 b_1, a_2 \wedge_2 b_2\rangle\,.$$

And define

$$\begin{pmatrix}\vee_2\\ \vee_1\end{pmatrix} : L_1 \times L_2 \longrightarrow L_1 \times L_2$$

such that

$$\langle a_1, a_2\rangle \begin{pmatrix}\vee_2\\ \vee_1\end{pmatrix} \langle b_1, b_2\rangle = \langle a_1 \vee_1 b_1, a_2 \vee_2 b_2\rangle\,.$$

Then

$$\left\langle L_1 \times L_2, \begin{pmatrix}\wedge_2\\ \wedge_1\end{pmatrix}, \begin{pmatrix}\vee_2\\ \vee_1\end{pmatrix}\right\rangle$$

is the coordinatewise product lattice of $\langle L_1, \wedge_1, \vee_1\rangle$ and $\langle L_2, \wedge_2, \vee_2\rangle$ and is the algebraic version of $\langle L_1 \times L_2, \precsim\rangle$, see [Davey and Priestley, 2002, p. 42].

Suppose as above that $\langle L_1, \leq_1\rangle$ and $\langle L_2, \leq_2\rangle$ are lattices. Let $\trianglelefteq$ be the narrowness-relation with respect to $\leq_1$ and $\leq_2$, i.e. for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in L_1 \times L_2$

$$\langle a_1, a_2\rangle \trianglelefteq \langle b_1, b_2\rangle \quad \text{iff} \quad b_1 \leq_1 a_1 \quad \text{and} \quad a_2 \leq_2 b_2.$$

It can be shown that $\langle L_1 \times L_2, \trianglelefteq\rangle$ is a lattice. Let

$$\langle L_1, \wedge_1, \vee_1\rangle \text{ and } \langle L_2, \wedge_2, \vee_2\rangle$$

be the algebraic formulation of $\langle L_1, \leq_1\rangle$ and $\langle L_2, \leq_2\rangle$ respectively. Define

$$\begin{pmatrix}\wedge_2\\ \vee_1\end{pmatrix} : L_1 \times L_2 \longrightarrow L_1 \times L_2$$

such that

$$\langle a_1, a_2\rangle \begin{pmatrix}\wedge_2\\ \vee_1\end{pmatrix} \langle b_1, b_2\rangle = \langle a_1 \vee_1 b_1, a_2 \wedge_2 b_2\rangle .$$

And define

$$\begin{pmatrix}\vee_2\\ \wedge_1\end{pmatrix} : L_1 \times L_2 \longrightarrow L_1 \times L_2$$

such that

$$\langle a_1, a_2\rangle \begin{pmatrix}\vee_2\\ \wedge_1\end{pmatrix} \langle b_1, b_2\rangle = \langle a_1 \wedge_1 b_1, a_2 \vee_2 b_2\rangle .$$

Then

$$\left\langle L_1 \times L_2, \begin{pmatrix}\wedge_2\\ \vee_1\end{pmatrix}, \begin{pmatrix}\vee_2\\ \wedge_1\end{pmatrix}\right\rangle$$

is a lattice and is the algebraic version of $\langle L_1 \times L_2, \trianglelefteq\rangle$.

3.6 The structure on minimal joinings

The next theorem gives a characterization of the structure, with respect to the lowerness-relation, of the elements in a joining-space that are maximally narrow, i.e., those called minimal joinings. Note that with $\min J$ is meant $\min_{\trianglelefteq} J$.

We recall the definition 3.4 on page 567 of a complete quasi-lattice.

Theorem 3.34 *Suppose that $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ is a* Js *and that $\mathcal{A}_1$ and $\mathcal{A}_2$ are complete quasi-lattices and denote the relation $\precsim / \min J$ as $\precsim^*$. Let $X \subseteq \min J$. Then*

(i) $\mathrm{lub}_{\precsim^*} X \neq \varnothing$ *and* $\mathrm{glb}_{\precsim^*} X \neq \varnothing$

(ii) *if $X \neq \varnothing$ then $\pi_2 \left[\mathrm{lub}_{\precsim^*} X\right] \subseteq \mathrm{lub}_{R_2} \pi_2 [X]$*

(iii) *if $X \neq \varnothing$ then $\pi_1 \left[\mathrm{glb}_{\precsim^*} X\right] \subseteq \mathrm{glb}_{R_1} \pi_1 [X]$.*

Proof. Suppose that $X \subseteq \min J$. Note that since $\mathcal{A}_1 = \langle A_1, R_1\rangle$ and $\mathcal{A}_2 = \langle A_2, R_2\rangle$ are complete quasi-lattices, then $\mathrm{glb}_{R_1} \pi_1 [X] \neq \varnothing$ and $\mathrm{lub}_{R_2} \pi_2 [X] \neq \varnothing$.

(I) We prove (iii). Suppose that $X \neq \varnothing$. From (1) in Theorem 3.30 it follows that there is $c_2 \in A_2$ such that if $a_1 \in \mathrm{glb}_{R_1} \pi_1 [X]$, $\langle a_1, c_2\rangle \in \min J$, and, furthermore, it holds that $c_2R_2a_2$ for all $a_2 \in \mathrm{glb}_{R_2}\,\pi_2\,[X]$. We shall now show that

$$\langle a_1, c_2\rangle \in \mathrm{glb}_{\precsim^*} X.$$

Suppose that $\langle x_1, x_2\rangle \in X$. Hence, $x_1 \in \pi_1\,[X]$ and $x_2 \in \pi_2\,[X]$. Since $a_1 \in \mathrm{glb}_{R_1}\,\pi_1\,[X]$, it follows that $a_1R_1x_1$. Suppose that $a_2 \in \mathrm{glb}_{R_2}\,\pi_2\,[X]$. Then $a_2R_2x_2$ and since $c_2R_2a_2$ it follows that $c_2R_2x_2$. From $a_1R_1x_1$ and $c_2R_2x_2$ follows that $\langle a_1, c_2\rangle \precsim \langle x_1, x_2\rangle$ and since $\langle a_1, c_2\rangle, \langle x_1, x_2\rangle \in \min J$ it follows that

$$\langle a_1, c_2\rangle \precsim^* \langle x_1, x_2\rangle\,.$$

Since $\langle x_1, x_2\rangle$ is an arbitrary element in X, it follows that

$$\langle a_1, c_2\rangle \in \mathrm{lb}_{\precsim^*} X.$$

Suppose now that $\langle y_1, y_2\rangle \in \min J$ and $\langle y_1, y_2\rangle \in \mathrm{lb}_{\precsim^*} X$. We shall prove that

$$\langle y_1, y_2\rangle \precsim^* \langle a_1, c_2\rangle\,.$$

Suppose $z_1 \in \pi_1\,[X]$. Then there is $z_2 \in \pi_2\,[X]$ such that $\langle z_1, z_2\rangle \in X$ and hence $\langle y_1, y_2\rangle \precsim^* \langle z_1, z_2\rangle$, which implies that $y_1R_1z_1$. Thus $y_1 \in \mathrm{lb}_{R_1}\,\pi_1\,[X]$ and since $a_1 \in \mathrm{glb}_{R_1}\,\pi_1\,[X]$, it follows that $y_1R_1a_1$. Since

$$\langle a_1, c_2\rangle, \langle y_1, y_2\rangle \in \min J \quad \text{and} \quad y_1R_1a_1$$

it follows from (ii) in Theorem 3.29 that $y_2R_2c_2$, which implies that

$$\langle y_1, y_2\rangle \precsim^* \langle a_1, c_2\rangle\,.$$

This shows that $\langle a_1, c_2\rangle \in \mathrm{glb}_{\precsim^*} X$ and hence $\mathrm{glb}_{\precsim^*} X \neq \varnothing$. Note that $a_1 \in \pi_1\left[\mathrm{glb}_{\precsim^*} X\right]$ and $a_1 \in \mathrm{glb}_{R_1}\,\pi_1\,[X]$. Suppose that $x_1 \in \pi_1\left[\mathrm{glb}_{\precsim^*} X\right]$. Then there is x_2 such that $\langle x_1, x_2\rangle \in \mathrm{glb}_{\precsim^*} X$. Since $\langle a_1, c_2\rangle \in \mathrm{glb}_{\precsim^*} X$ it follows that

$$\langle x_1, x_2\rangle \sim^* \langle a_1, c_2\rangle$$

which implies $x_1Q_1a_1$. Since $a_1 \in \mathrm{glb}_{R_1}\,\pi_1\,[X]$ it follows that $x_1 \in \mathrm{glb}_{R_1}\,\pi_1\,[X]$. This shows that

$$\pi_1\left[\mathrm{glb}_{\precsim^*} X\right] \subseteq \mathrm{glb}_{R_1}\,\pi_1\,[X]\,.$$

(II) The proof of (ii) is analogous with the proof of (iii).

(III) That (i) holds when $X \neq \varnothing$ follows from the proof of (ii) and (iii). The proof that $\mathrm{lub}_{\precsim^*} \varnothing \neq \varnothing$ and $\mathrm{glb}_{\precsim^*} \varnothing \neq \varnothing$ follows from the lemma below. (To see this, cf. as well the remark above Theorem 3.5.) ■

Lemma 3.35 *Suppose that $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ is a non-empty joining-system and that $\mathcal{A}_1 = \langle A_1, R_1\rangle$ and $\mathcal{A}_2 = \langle A_2, R_2\rangle$ are complete quasi-lattices. Then*

(i) *there are $a_1 \in \mathrm{lub}_{R_1}\, \pi_1\,[J]$ and $a_2 \in \mathrm{glb}_{R_2}\, J\,[a_1]$ and the following holds: $\langle a_1, a_2\rangle \in \min J$ and $\langle a_1, a_2\rangle$ is a greatest element in $\min J$ with respect to $\precsim$.*

(ii) *there are $b_2 \in \mathrm{glb}_{R_2}\, \pi_2\,[J]$ and $b_1 \in \mathrm{lub}_{R_1}\, J^{-1}\,[b_2]$ and the following holds: $\langle b_1, b_2\rangle \in \min J$ and $\langle b_1, b_2\rangle$ is a least element in $\min J$ with respect to $\precsim$.*

Proof. (I) We prove (i). Since $\mathcal{A}_i$ $(i = 1, 2)$ is a complete quasi-lattice, there is $g_i \in A_i$ such that g_i is a greatest element in $\mathcal{A}_i$ with respect to R_i and $l_i \in A_i$ such that l_i is a least element in $\mathcal{A}_i$. According to the assumption, $J \neq \varnothing$. Suppose that $\langle x_1, x_2\rangle \in J$. Note that $x_2R_2g_2$ and from condition (1) in the definition of a joining-system follows $\langle x_1, g_2\rangle \in J$. Since $\mathcal{A}_1$ is a complete quasi-lattice it follows that $\mathrm{lub}_{R_1}\, \pi_1\,[J] \neq \varnothing$. Suppose that $a_1 \in \mathrm{lub}_{R_1}\, \pi_1\,[J]$. From condition (2) of a joining-system follows that $\langle a_1, g_2\rangle \in J$. Since $\mathcal{A}_2$ is a complete quasi-lattice $\mathrm{glb}_{R_2}\, J\,[a_1] \neq \varnothing$. Suppose that $a_2 \in \mathrm{glb}_{R_2}\, J\,[a_1]$. Then $\langle a_1, a_2\rangle \in J$ according to condition (3) of a joining-system. Suppose that $\langle y_1, y_2\rangle \in J$ and $\langle y_1, y_2\rangle \lhd \langle a_1, a_2\rangle$. Then

$$(*)\ a_1R_1y_1 \& y_2P_2a_2$$

or

$$(**)\ a_1P_1y_1 \& y_2R_2a_2$$

Since $y_1 \in \pi_1\,[J]$ and $a_1 \in \mathrm{lub}_{R_1}\, \pi_1\,[J]$ it follows that $y_1R_1a_1$ and therefore $(**)$ above does not hold. $a_1R_1y_1$ implies $y_1Q_1a_1$ and hence $y_2 \in J\,[a_1]$. Since $a_2 \in \mathrm{glb}_{R_2}\, J\,[a_1]$ it follows that $a_2R_2y_2$. This shows that $(*)$ above does not hold. Thus $\langle a_1, a_2\rangle \in \min J$. Suppose that $\langle z_1, z_2\rangle \in \min J$. Then $z_1 \in \pi_1\,[J]$ and since $a_1 \in \mathrm{lub}_{R_1}\, \pi_1\,[J]$ it follows that $z_1R_1a_1$ and thus $\langle z_1, z_2\rangle \precsim \langle a_1, a_2\rangle$.

(II) The proof of (ii) is analogous with the proof of (i). ■

Corollary 3.36 *Given the assumption in Theorem 3.34, $\langle \min J, \precsim^*\rangle$ is a complete quasi-lattice.*

The theorem 3.37 below is a kind of converse of the theorem 3.34 above. We recall that $\uparrow K$ is the enclosure of K (see definition 3.23 above on page 578).

Theorem 3.37 *Suppose that $\mathcal{A}_1 = \langle A_1, R_1\rangle$ and $\mathcal{A}_2 = \langle A_2, R_2\rangle$ are quasi-orderings and $K \subseteq A_1 \times A_2$ is such that for all $\langle a_1, a_2\rangle \in K$, $\langle a_1, a_2\rangle$ is a minimal element in K with respect to $\trianglelefteq$. Suppose further that $\precsim_K$ is the relation $\precsim$ on $A_1 \times A_2$ restricted to K and that $\langle K, \precsim_K\rangle$ is a complete quasi-lattice and the following two conditions hold:*

(i) For all $X \subseteq K, \pi_2 \left[\mathrm{lub}_{\precsim_K} X\right] \subseteq \mathrm{lub}_{R_2} \pi_2 [X]$.

(ii) For all $X \subseteq K, \pi_1 \left[\mathrm{glb}_{\precsim_K} X\right] \subseteq \mathrm{glb}_{R_1} \pi_1 [X]$.

Then $\langle \mathcal{A}_1, \mathcal{A}_2, \uparrow K\rangle$ is a joining-system and $\min \uparrow K = K$.

Proof. (I) Proof of condition (1) in the definition of a joining-system. Suppose that $\langle a_1, a_2\rangle \in \uparrow K$ and $\langle a_1, a_2\rangle \trianglelefteq \langle b_1, b_2\rangle$. Then there is $\langle c_1, c_2\rangle \in K$ such that $\langle c_1, c_2\rangle \trianglelefteq \langle a_1, a_2\rangle$, and it follows that $\langle c_1, c_2\rangle \trianglelefteq \langle b_1, b_2\rangle$, which implies that $\langle b_1, b_2\rangle \in \uparrow K$.

(II) Proof of condition (2) in the definition of a joining-system. Suppose that $C_1 \subseteq A_1$, $b_2 \in A_2$ and that $a_1 \in \mathrm{lub}_{R_1} C_1$. Suppose further that for all $c_1 \in C_1$, $\langle c_1, b_2\rangle \in \uparrow K$. We show that $\langle a_1, b_2\rangle \in \uparrow K$. For all $c_1 \in C_1$, there is an element $\langle c_1^*, b_2^{c_1}\rangle \in K$ such that $\langle c_1^*, b_2^{c_1}\rangle \trianglelefteq \langle c_1, b_2\rangle$. Since $\langle K, \precsim_K\rangle$ is a complete quasi-lattice it follows that there is $\langle x_1, x_2\rangle \in K$ such that

$$(***)\ \ \langle x_1, x_2\rangle \in \mathrm{lub}_{\precsim_K} \left\{\langle c_1^*, b_2^{c_1}\rangle \mid c_1 \in C_1\right\}.$$

Hence,

$$x_2 \in \pi_2 \left[\mathrm{lub}_{\precsim_K} \left\{\langle c_1^*, b_2^{c_1}\rangle \mid c_1 \in C_1\right\}\right].$$

From the assumption (i) follows that

$$x_2 \in \mathrm{lub}_{R_2} \pi_2 \left[\left\{\langle c_1^*, b_2^{c_1}\rangle \mid c_1 \in C_1\right\}\right]$$

and hence

$$x_2 \in \mathrm{lub}_{R_2} \left\{b_2^{c_1} \mid c_1 \in C_1\right\}.$$

Note that

$$b_2 \in \mathrm{ub}_{R_2} \left\{b_2^{c_1} \mid c_1 \in C_1\right\}$$

which implies that $x_2 R_2 b_2$.

From $(***)$ above it follows that for all $c_1 \in C_1$

$$\langle c_1^*, b_2^{c_1}\rangle \precsim_K \langle x_1, x_2\rangle$$

and hence $c_1^* R_1 x_1$. For all $c_1 \in C_1$

$$\langle c_1^*, b_2^{c_1}\rangle \trianglelefteq \langle c_1, b_2\rangle$$

which implies $c_1 R_1 c_1^*$ and hence $c_1 R_1 x_1$. Thus $x_1 \in \mathrm{ub}_{R_1} C_1$ and since $a_1 \in \mathrm{lub}_{R_1} C_1$ it follows that $a_1 R_1 x_1$. This together with $\langle x_1, x_2 \rangle \in K$ and $x_2 R_2 b_2$ implies (see part (I) in this proof) $\langle a_1, b_2 \rangle \in \uparrow K$.

(III) Proof of condition (3) in the definition of a joining-system is analogous to the proof of condition (2) in (II).

(IV) Proof of $\min \uparrow K = K$. Suppose that $\langle a_1, a_2 \rangle \in K$ and show that $\langle a_1, a_2 \rangle \in \min \uparrow K$. Suppose that $\langle b_1, b_2 \rangle \in \uparrow K$ such that $\langle b_1, b_2 \rangle \trianglelefteq \langle a_1, a_2 \rangle$. Since $\langle b_1, b_2 \rangle \in \uparrow K$ there is $\langle c_1, c_2 \rangle \in K$ such that $\langle c_1, c_2 \rangle \trianglelefteq \langle b_1, b_2 \rangle$. Hence, $\langle c_1, c_2 \rangle \trianglelefteq \langle a_1, a_2 \rangle$ and since $\langle a_1, a_2 \rangle, \langle c_1, c_2 \rangle \in K$ and all elements in K are minimal elements in K with respect to $\trianglelefteq$, it follows that $\langle a_1, a_2 \rangle \backsimeq \langle c_1, c_2 \rangle$, which implies that $\langle a_1, a_2 \rangle \backsimeq \langle b_1, b_2 \rangle$ and $\langle a_1, a_2 \rangle \in \min \uparrow K$.

Suppose that $\langle a_1, a_2 \rangle \in \min \uparrow K$. Then $\langle a_1, a_2 \rangle \in \uparrow K$ and there is $\langle b_1, b_2 \rangle \in K$ such that $\langle b_1, b_2 \rangle \trianglelefteq \langle a_1, a_2 \rangle$. According to what have just been proven, from $\langle b_1, b_2 \rangle \in K$ follows that $\langle b_1, b_2 \rangle \in \min \uparrow K$. This implies that $\langle b_1, b_2 \rangle \backsimeq \langle a_1, a_2 \rangle$, and thus $\langle a_1, a_2 \rangle \in K$. ■

3.7 Networks of joining-systems

A normative system is not always represented by just one joining-system. More complex normative systems are usually represented by a network of joining-systems. (A rudimentary network is shown in Section 5.2.3.) In such representations, the relative product of joining spaces is an important operation for the construction of new joining-systems. The theorem below describes the situation.

Note that, when more than two joining-systems are involved, the sign J for a set of joinings will be annexed with two indices. Thus, the set of joinings from a quasi-ordering $\mathcal{A}_i$ to a quasi-ordering $\mathcal{A}_j$ will be denoted $J_{i,j}$. Accordingly, the joining-system from $\mathcal{A}_i$ to $\mathcal{A}_j$ is denoted $\langle \mathcal{A}_i, \mathcal{A}_j, J_{i,j} \rangle$.

Theorem 3.38 *Suppose that $\langle \mathcal{A}_1, \mathcal{A}_2, J_{1,2} \rangle$ and $\langle \mathcal{A}_2, \mathcal{A}_3, J_{2,3} \rangle$ are joining-systems and that $\mathcal{A}_2$ is a complete quasi-lattice. Then $\langle \mathcal{A}_1, \mathcal{A}_3, J_{1,2} | J_{2,3} \rangle$ is a joining-system and is called the relative product of $\langle \mathcal{A}_1, \mathcal{A}_2, J_{1,2} \rangle$ and $\langle \mathcal{A}_2, \mathcal{A}_3, J_{2,3} \rangle$.*

Proof. We begin by proving condition (1) in the definition of a *Js* (Definition 3.11 in Section 3.2.2). Suppose that $\langle a_1, a_3 \rangle \in J_{1,2} | J_{2,3}$ and $\langle a_1, a_3 \rangle \trianglelefteq \langle b_1, b_3 \rangle$. From $\langle a_1, a_3 \rangle \in J_{1,2} | J_{2,3}$ follows that there is $a_2 \in A_2$ such that $\langle a_1, a_2 \rangle \in J_{1,2}$ and $\langle a_2, a_3 \rangle \in J_{2,3}$. From $\langle a_1, a_3 \rangle \trianglelefteq \langle b_1, b_3 \rangle$ follows that $b_1 R_1 a_1$ and $a_3 R_3 b_3$. Since $\langle \mathcal{A}_1, \mathcal{A}_2, J_{1,2} \rangle$ is a joining-system, $b_1 R_1 a_1$ and $\langle a_1, a_2 \rangle \in J_{1,2}$ implies that $\langle b_1, a_2 \rangle \in J_{1,2}$. And $a_3 R_3 b_3$ and $\langle a_2, a_3 \rangle \in J_{2,3}$ implies that $\langle a_2, b_3 \rangle \in J_{2,3}$, since $\langle \mathcal{A}_2, \mathcal{A}_3, J_{2,3} \rangle$ is a joining-system. From $\langle b_1, a_2 \rangle \in J_{1,2}$ and $\langle a_2, b_3 \rangle \in J_{2,3}$ follows that $\langle b_1, b_3 \rangle \in J_{1,2} | J_{2,3}$.

We now prove condition (2) in the definition of a *Js*. Suppose that $C_1 \subseteq A_1$ and $C_1 \neq \varnothing$ such that for all $c_1 \in C_1$, $\langle c_1, b_3 \rangle \in J_{1,2}|J_{2,3}$ and suppose $a_1 \in \mathrm{lub}_{R_1} C_1$. Let

$$C_1^{(2)} = \{c_2 \in A_2 \mid \exists c_1 \in C_1 : \langle c_1, c_2 \rangle \in J_{1,2} \ \&\ \langle c_2, b_3 \rangle \in J_{2,3}\}$$

Hence, for all $c_2 \in C_1^{(2)}$, $\langle c_2, b_3 \rangle \in J_{2,3}$. Since $\mathcal{A}_2$ is a complete quasi-lattice (Definition 3.4), it follows that $\mathrm{lub}_{R_2} C_1^{(2)} \neq \varnothing$. Suppose that $a_2 \in \mathrm{lub}_{R_2} C_1^{(2)}$. Since $\langle \mathcal{A}_2, \mathcal{A}_3, J_{2,3} \rangle$ is a *Js* it follows that $\langle a_2, b_3 \rangle \in J_{2,3}$. For all $c_1 \in C_1$, there is $c_1^{(2)} \in C_1^{(2)}$ such that $\left\langle c_1, c_1^{(2)} \right\rangle \in J_{1,2}$. Since $\langle \mathcal{A}_1, \mathcal{A}_2, J_{1,2} \rangle$ is a *Js*, this implies that $\langle c_1, a_2 \rangle \in J_{1,2}$ for all $c_1 \in C_1$, and, consequently, $\langle a_1, a_2 \rangle \in J_{1,2}$. Since $\langle a_2, b_3 \rangle \in J_{2,3}$ it follows that $\langle a_1, b_3 \rangle \in J_{1,2}|J_{2,3}$.

The proof of condition (3) is analogous and is omitted. ■

Note that from the assumption $J_{1,2}|J_{2,3} = J_{1,3}$ and the requirement of connectivity it follows that $\min J_{1,2}| \min J_{2,3} \subseteq \min J_{1,3}$. Also, however, note that $\subseteq$ cannot generally be strengthened to $=$ (Cf. [Lindahl and Odelstad, 2011, sect. 3.3.2]).

3.8 Intervenients

The notion of "intervenient" (cf. above, Section 2.2) will be treated in detail in Section 5, in connection with Boolean quasi-orderings and Boolean joining-systems. As a general notion, it is, however, introduced here.

Let us consider three joining-systems

$$\mathcal{S}_1 = \langle \mathcal{A}_1, \mathcal{A}_2, J_{1,2} \rangle, \mathcal{S}_2 = \langle \mathcal{A}_2, \mathcal{A}_3, J_{2,3} \rangle,\ \mathcal{S}_3 = \langle \mathcal{A}_1, \mathcal{A}_3, J_{1,3} \rangle,$$

where $\mathcal{A}_i = \langle A_i, R_i \rangle$. There can be $a_1 \in A_1$, $a_2 \in A_2$, and $a_3 \in A_3$ such that $\langle a_1, a_2 \rangle \in J_{1,2}$, $\langle a_2, a_3 \rangle \in J_{2,3}$, and $\langle a_1, a_3 \rangle \in J_{1,3}$. A case of special interest then, is when $\mathrm{WG}_{\mathcal{S}_1}(a_1, a_2, A_1)$ and $\mathrm{SC}_{\mathcal{S}_2}(a_3, a_2, A_3)$, i.e., when, in $\mathcal{S}_1$, a_1 is among the weakest grounds in A_1 for a_2, and a_3 is among the strongest consequences in A_3 of a_2. (Cf. above, Section 3.3). In this case, a_2, in a sense, is "intermediate" between a_1 and a_3 and "mediates" the joining $\langle a_1, a_3 \rangle$. Therefore, in this case we call a_2 an *intervenient*.

In order to give a more detailed formal exposition of what is said above, we first give the following definition of a *simple Js-triple*.

Definition 3.39 *Suppose that* $\mathcal{S}_1 = \langle \mathcal{A}_1, \mathcal{A}_2, J_{1,2} \rangle$, $\mathcal{S}_2 = \langle \mathcal{A}_2, \mathcal{A}_3, J_{2,3} \rangle$ *and* $\mathcal{S}_3 = \langle \mathcal{A}_1, \mathcal{A}_3, J_{1,3} \rangle$ *are joining-systems where* $\mathcal{A}_i = \langle A_i, R_i \rangle$. $\langle \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \rangle$

is a simple Js-triple *if* A_1, A_2 *and* A_3 *are pair-wise disjunct, and, for the relative product* $J_{1,2}|J_{2,3}$ *it holds that* $J_{1,3} = J_{1,2}|J_{2,3}$.[19]

(For *Bjs-triples* of Boolean joining-systems, cf. Section 5.1.)

Then the notion of *intervenient* in a simple *Js*-triple is defined as follows.

Definition 3.40 *In a simple* Js-*triple* $\langle \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \rangle$*, the element* $a_2 \in A_2$*, is an* intervenient from $\mathcal{A}_1$ to $\mathcal{A}_3$ corresponding to *the joining* $\langle a_1, a_3 \rangle \in J_{1,3}$*, denoted* $a_2 \curvearrowright \langle a_1, a_3 \rangle$ *, if* a_1 *is a weakest ground of* a_2 *in* $\mathcal{S}_1$ *and* a_3 *is a strongest consequence of* a_2 *in* $\mathcal{S}_3$*.*

Since weakest grounds and strongest consequences are related to minimal joinings, the same holds for intervenients. If a_2 is an intervenient corresponding to $\langle a_1, a_3 \rangle$, there is $b_2 \in A_2$ such that $\langle a_1, b_2 \rangle$ is a minimal joining and $b_2 R_2 a_2$. And, further, there is $c_2 \in A_2$ such that $\langle c_2, a_3 \rangle$ is a minimal joining and $a_2 R_2 c_2$. If $\langle a_1, a_2 \rangle$ is a minimal element, then, since a_2 is minimal with respect to the ground a_1, a_2 is called ground-minimal. If $\langle a_2, a_3 \rangle$ is a minimal element, then, since a_2 is minimal with respect to the consequence a_3, a_2 is called consequence-minimal. A very convenient way of representing a normative system is if all intervenients are ground- and consequence-minimal and the operation relative product is used. Changes of the normative system are then simplified and the notion of open intermediate concepts is elucidated.

A step towards analyzing more general structures in the law is taking into account chains of four or more quasi-orderings. Let us pay regard to joining-systems involving four quasi-orderings $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ such that $a_2 \curvearrowright \langle a_1, a_3 \rangle$ and $a_3 \curvearrowright \langle a_2, a_4 \rangle$. (See Figure 13.) From this follows that $\mathrm{WG}(a_2, a_3, A_2)$ & $\mathrm{SC}(a_3, a_2, A_3)$. This conjunction is equivalent to $\langle a_2, a_3 \rangle \in \min J_{2,3}$, see Theorem 3.21. (This is illustrated by the thick line in Figure 13.) Note that a chain of four quasi-orderings can be continued at any length by adding $\mathcal{A}_5, \mathcal{A}_6$, and so on. The notion of intervenient is of particular interest when the three joining-systems are Boolean joining-systems. This will be the subject-matter of the subsequent Section 5, where conjunctions, disjunctions and negations of intervenients are studied, organic wholes of intervenients discussed and a typology of intervenients presented. Also, section 5 will contain several examples of legal intervenients.

[19] The triple is simple in the following sense. The presupposition of disjunct strata will make it possible in the present section to disregard the problem with "degenerated" weakest grounds and/or strongest consequences. This problem will be dealt with in connection with intervenients in Boolean joining systems.

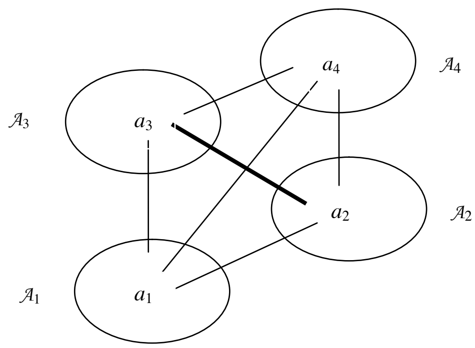


Figure 13

4 TJS for Boolean joining-systems

In the representation of a normative system, the connectives "and", "or" and "not" are often essential. This is neatly illustrated in the example of Amendment XIV in the U.S. Constitution, quoted above (Section 1.7.1):

> "All persons born *or* naturalized in the United States, *and* subject to the jurisdiction thereof, are citizens of the United States *and* of the State wherein they reside. *No* State shall make *or* enforce any law which shall abridge the privileges *or* immunities of citizens of the United States; *nor* shall any State deprive any person of life, liberty, *or* property, without due process of law; *nor* deny to any person within its jurisdiction the equal protection of the laws."

With a view to the connectives referred to, in the present Section 4 and the subsequent Section 5, we consider strata of *Boolean quasi-orderings* (*Bqo*'s) and joining-systems that are *Boolean joining-systems* (*Bjs*'). As mentioned, the development of TJS for *Bqo*'s and *Bjs*'s in this chapter of the Handbook relies much on earlier papers by the present authors and the reader will often be referred to these papers for further details and for proofs of the results.

4.1 Boolean quasi-orderings and Boolean joining-systems

4.1.1 Boolean quasi-orderings

The notion of Boolean quasi-ordering is defined as follows.

Definition 4.1 *The relational structure* $\mathcal{B} = \langle B, \wedge, ', R\rangle$ *is a* Boolean quasi-ordering (Bqo) *if* $\langle B, \wedge, '\rangle$ *is a Boolean algebra and* R *is a quasi-ordering,* $\perp$ *is the zero element and* $\top$ *is the unit element, such that* R *satisfies the additional requirements:*

(1) aRb *and* aRc *implies* $aR(b \wedge c)$*,*

(2) aRb *implies* $b'Ra'$*,*

(3) $(a \wedge b)Ra$*,*

(4) not $\top R \perp$*.*

Note that if $\leq$ is the partial ordering determined by $\langle B, \wedge, '\rangle$, from requirement (3) it follows that $a \leq b$ implies aRb. As usual, $\leq$ is defined by $a \leq b$ if and only if $a \wedge b = a$.

Requirements (3) and (4) can be expressed equivalently by saying that R is a non-total super-relation of the Boolean ordering $\leq$. More exactly, suppose that $\langle B, \wedge, '\rangle$ is a Boolean algebra, that $\leq$ is the partial ordering determined by the algebra, and that R is a transitive relation on B. Then the conjunction of (3) and (4) is equivalent to the conjunction of (i) $\leq$ is a subset of R, and (ii) R is a proper subset of $B \times B$.

Some general notions relating to *Bqo*'s are as follows (see [Lindahl and Odelstad, 2004, sect. 2.1]):

If $\langle B, \wedge, ', R\rangle$ is a *Bqo* then we say that the Boolean algebra $\langle B, \wedge, '\rangle$ is the *reduct* of $\langle B, \wedge, ', R\rangle$. In what follows, the reduct $\langle B, \wedge, '\rangle$ of a *Bqo* $\mathcal{B}$ will be denoted $\mathcal{B}^{red}$. Suppose that $\mathcal{B} = \langle B, \wedge, ', R\rangle$ is a *Bqo* and Q is the indifference part of R. The *quotient algebra* of $\mathcal{B}$ with respect to Q is a structure $\langle B/Q, \cap, -, \leq_Q\rangle$ such that $\langle B/Q, \cap, -\rangle$ is a Boolean algebra and $\leq_Q$ is the partial ordering determined by this algebra. The natural mapping of $\langle B, \wedge, '\rangle$ onto $\langle B/Q, \cap, -\rangle$ is a homomorphism (cf. [Odelstad and Lindahl, 2000]). We call $\langle B/Q, \cap, -\rangle$ the *quotient reduction* of $\mathcal{B}$. Thus there are two Boolean algebras which should be kept apart, namely $\mathcal{B}^{red}$, i.e. the reduct of $\mathcal{B}$, and the quotient reduction of $\mathcal{B}$. If the quotient reduction of $\mathcal{B}$ is isomorphic to $\mathcal{B}^{red}$, $R = \leq$, and we say that $\mathcal{B}$ is *conservatively reducible.*

As just mentioned, the transition to the quotient algebra of $\langle B, \wedge, ', R\rangle$ with respect to the equality part Q of R will result in a new Boolean algebra. In what follows we will not make this transition. The point is that, in the models we have in mind, even though, for a and b it holds that aQb (and therefore a and b belong to the same Q-equivalence class), we may want to distinguish a and b because they can have different meaning. We get possibilities of finer divisions when we can distinguish the three possibilities: 1. $a = b$, 2. $a \neq b$ and aQb, 3. $a \neq b$ and not aQb. Therefore, there is a point in remaining within the framework of Boolean quasi-orderings as defined above.

Note that if $\mathcal{B} = \langle B, \wedge,', R\rangle$ is a *Bqo*, then

$$(a \vee b) \in \mathrm{lub}_R \{a, b\},$$
$$(a \wedge b) \in \mathrm{glb}_R \{a, b\}.$$

If $\mathcal{B} = \langle B, \wedge,', R\rangle$ is a *Bqo*, then $\langle B, R\rangle$ is a quasi-ordering and, of course, what is said about quasi-orderings in section 3 is applicable to $\mathcal{B}$. We say that the *Bqo* $\langle B, \wedge,', R\rangle$ is *complete* if the quasi-ordering $\langle B, R\rangle$ is a complete quasi-lattice.

4.1.2 Boolean joining systems

A fundamental construction for the representation of a normative system is that of a Boolean joining-system. If $\mathcal{N}$ is a two-strata system of conditional norms, then $\mathcal{N}$ can be represented by a *Bjs* $\langle \mathcal{B}_1, \mathcal{B}_2, J\rangle$ where J is a set of conditional norms, where $\mathcal{B}_1$ is a *Bqo* of grounds, and $\mathcal{B}_2$ is a *Bqo* of normative consequences.

Definition 4.2 $\langle \mathcal{B}_1, \mathcal{B}_2, J\rangle$ *is a* Boolean joining system (Bjs) *if*

$$\mathcal{B}_1 = \langle B_1, \wedge,', R_1\rangle,\ \mathcal{B}_2 = \langle B_2, \wedge,', R_2\rangle$$

are Boolean quasi-orderings and $\langle \langle B_1, R_1\rangle, \langle B_2, R_2\rangle, J\rangle$ *is a joining-system.*

With the definition of a *Bjs* now given it is clear that the results for joining-systems in Section 3 apply to the *Bjs* version of joining-systems. This holds e.g., for the notions of weakest ground, strongest consequence, minimal joinings and connectivity.

In the study of *Bjs*'s, structures that are not *Bqo's* play an essential role. This is exemplified by the following theorem, which is proved in [Lindahl and Odelstad, 2011, p. 128].

Theorem 4.3 *Suppose that* $\langle \mathcal{B}_1, \mathcal{B}_2, J\rangle$ *is a* Bjs *that satisfies connectivity. Then* $\langle \min J, \precsim\rangle$ *is a quasi-lattice.*

Cf. Corollary 3.36 above.

If $\langle \mathcal{B}_1, \mathcal{B}_2, J\rangle$ is a Boolean joining system, it is often reasonable that falsum in $\mathcal{B}_1$ and in $\mathcal{B}_2$ are the same element $\bot$ and that the same holds for verum $\top$. From this follows that in J there are joinings, which are degenerated in the sense that they do not seem to fulfill the intuitive idea behind the notion of a joining, for example $\langle \bot, \bot\rangle$ and $\langle \top, \top\rangle$.

Referring to a *Bjs* $\langle \mathcal{B}_1, \mathcal{B}_2, J\rangle$, however, we introduce a distinction between "degenerated" and "non-degenerated" for weakest ground, strongest consequences and joining.

(1) If $\mathrm{WG}(\perp, a_2, \mathcal{B}_1)$, the weakest ground in B_1 for a_2 is degenerated; similarly, if $\mathrm{SC}(\top, a_1, \mathcal{B}_2\rangle$, the strongest consequence in B_2 of a_1 is degenerated.

(2) As joinings from $\mathcal{B}_1$ to $\mathcal{B}_2$, the elements in

$$\{\langle\perp,\perp\rangle, \langle\top,\top\rangle, \langle b_1,\top\rangle, \langle\perp, b_2\rangle\}$$

are degenerated joinings.

Note that $\langle\perp,\perp\rangle, \langle\top,\top\rangle \in J$, and even $\langle\perp,\perp\rangle, \langle\top,\top\rangle \in \min J$. Note further that if $b_2 \in B_2$ and there is no $b_1 \in B_1 \setminus \{\perp\}$ such that $\langle b_1, b_2\rangle \in J$, then $\langle\perp, b_2\rangle \in \min J$. Analogously, if $b_1 \in B_1$ and there is no $b_2 \in B_2 \setminus \{\top\}$ such that $\langle b_1, b_2\rangle \in J$, then $\langle b_1, \top\rangle \in \min J$.

4.2 The condition implication model (cis)

We recall the statement by [Alchourrón and Bulygin, 1971] (referred to in the introductory Section 1), that a set α of sentences deductively correlates a pair $\langle p, q\rangle$ of sentences if q is a deductive consequence of $\{p\} \cup \alpha$, (or, using the relation Cn of consequence, if $q \in Cn(\{p\} \cup \alpha)$.) Also, we recall our remark that if propositional logic is used as a basis, it is usually presupposed that p, q are closed sentences with no free variables,(i.e., for example, p is the sentence "Smith has promised to pay Jones \$100" and q is "Smith has an obligation to pay \$100 to Jones"). Thus, in such sentences, individuals are referred to by individual constants (names).

A sentence such as "Smith has an obligation to pay \$100 to Jones" is often said to express an "individual norm". Owing to its general character, the *Bjs* theory can be used for representing correlations of conditional individual norms and derivation of individual norms.

As mentioned in Section 1, however, a normative system usually expresses general rules where no individual names occur. If the task is to represent a normative system of this ordinary kind, the feature of generality has to be taken into account. What will here be called the theory of *condition implication structures* (*cis*'s) is a special variety of the *Bjs* theory where the elements of B in a *Bqo* $\langle B, \wedge, ', R\rangle$ are *conditions*.

In general terms, a *cis* is a structure $\langle C, \rightarrow\rangle$ where C is a set of *conditions* and $\rightarrow$ is an implicative relation. In what follows we have in view especially the case of a *cis-Bqo* $\langle B, \wedge, ', R\rangle$, where B is a set of conditions and R is the implicative relation. A *cis-Bjs* is a *Bjs* $\langle \mathcal{B}_1, \mathcal{B}_2, J\rangle$ where the *Bqo*'s $\mathcal{B}_1$ and $\mathcal{B}_2$ are *cis*'. Part of a normative system can often be represented by a *cis-Bjs* $\langle \mathcal{B}_1, \mathcal{B}_2, J\rangle$ where $\mathcal{B}_1, \mathcal{B}_2$ are *cis*', and J is a correspondence from the set B_1 of conditions to the set B_2 of conditions.

In simple cases, conditions can be denoted by expressions using the sign of the infinitive, such as "to be 21 years old", "to be a citizen of the U.S.", "to be a child of", "to be entitled to inherit", or by corresponding expressions in the ing-form, like "being 21 years old" etc. Often, however, conditions should appropriately be expressed by open sentences, like "x promises to pay \$$y$ to z", "x is a citizen of state y", "x is entitled to inherit y".

When a condition is expressed by an open sentence, free variables like $x, y, z, ...$ occurring in the sentence merely are place-holders for expressing the condition in a convenient way and keeping track of the order of the places. In simple cases like, "committing murder implies being liable to imprisonment", place-holders are not needed. For details about Boolean operations on conditions, the reader is referred to [Lindahl and Odelstad, 2004, sect. 3].

In a *cis-Bqo* $\langle B, \wedge, ', R\rangle$, a condition a in B, such as "x promises to pay \$$y$ to z ", is said to be *fulfilled* or *non-fulfilled* by a particular triple, like ⟨Smith,100,Jones⟩. The fulfillment of a condition by a particular $n-$tuple of individuals is expressed by a closed sentence naming the individuals of the $n-$tuple.

A framework with implication between conditions seems to accord with the presupposed ontology of legal language, where terms such as "citizenship", "inheritance", "ownership", denote conditions that are treated as objects between which there is an implicative relation of "ground-consequence", often expressed in terms of "gives rise to" or "causes", or "implies". Thus inheritance is said to give rise to ownership, and ownership is said to imply a bundle of liberties, claims, and immunities.

Let us recall the remark after Definition 3.12 that if $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ is a joining-system, then $R_1|J|R_2 = J$ and, therefore, J can be said to "absorb" R_1 and R_2. From this it follows that if we have in view a *cis-Bjs* $\langle \mathcal{B}_1, \mathcal{B}_2, J\rangle$, where a_1, b_1, a_2, b_2 are conditions such that $a_1, b_1 \in B_1$ and $a_2, b_2 \in B_2$, we can use the following schema of derivation:

$$\begin{array}{l} (1)\ a_1 R_1 b_1 \\ (2)\ \langle b_1, a_2\rangle \in J \\ (3)\ a_2 R_2 b_2 \\ \hline (4)\ \langle a_1, b_2\rangle \in J \end{array}$$

In this schema, the joining (4) of two conditions is derived from the joining (2) together with implications (1) and (3).

4.2.1 A note on cis models with lattice-based quasi-orderings

Some kinds of conditions do not constitute Boolean algebras. One example is equality-relations. The term "equality-relation" here refer to a relation of equality with respect to some aspect α, and it is presupposed in this context that an equality-relation is always an equivalence-relation, i.e. a reflexive, transitive and symmetric relation. Let A be a non-empty set and let $E(A)$ be the set of equivalence relations on A. Define the binary relation $\leq$ on $E(A)$ in the following way: For all $\varepsilon_1, \varepsilon_2 \in E(A)$

$$\varepsilon_1 \leq \varepsilon_2 \text{ iff } x\varepsilon_1 y \text{ implies } x\varepsilon_2 y.$$

The reader should be reminded of the fact that $\mathcal{E}(A) = \langle E(A), \leq\rangle$ is a complete lattice. Note that the negation ε' of an equivalence relation $\varepsilon \in E(A)$ is not an equivalence relation, i.e. $\varepsilon' \notin E(A)$. $\langle E(A), \leq\rangle$, therefore, does not constitute a Boolean algebra. (Cf. [Odelstad, 2008, pp. 38f.].)

As appear from the foregoing, a Boolean quasi-ordering is a Boolean algebra extended with a quasi-ordering satisfying certain conditions. We can define an analogous structure based on a lattice instead of a Boolean algebra.

Definition 4.4 *The relational structure* $\langle L, \wedge, \vee, R\rangle$ *is a* lattice-based quasi-ordering (Lqo) *if* $\langle L, \wedge, \vee\rangle$ *is a lattice and* R *is a quasi-ordering such that* R *satisfies the additional requirements:*

(1) aRb *and* aRc *implies* $aR(b \wedge c)$,

(2) aRc *and* bRc *implies* $(a \vee b)Rc$,

(3) $(a \wedge b)Ra$,

(4) $aR(a \vee b)$.

The transition to the quotient algebra of $\langle L, \wedge, \vee\rangle$ with respect to the equality part of R will result in a lattice. (Cf. [Lindahl and Odelstad, 1999a, p. 171].) Let $\leq$ be the partial ordering determined by the lattice-based quasi-ordering $\langle L, \wedge, \vee, R\rangle$.[20] From requirement (3) for lattice-based quasi-orderings it follows that $a \leq b$ implies aRb. If $\langle A, \wedge, \vee, R\rangle$ is a lattice-based quasi-ordering then $\langle L, R\rangle$ is a quasi-lattice. Note that a *Bqo* determines a *Lqo*.

4.3 Subtraction and addition of norms: an example

In Section 1.6 above, we mentioned that TJS deals with subtraction and addition of norms in terms of the structure of the set $\min J$ of minimal joinings. In the present subsection we illustrate this issue by a *cis* concerning the legal effects of an illegal transfer of goods belonging to someone else. (Cf. [Lindahl and Odelstad, 2003].)

[20] As usual, $\leq$ is defined by $a \leq b$ if and only if $a \wedge b = a$.

Consider the following example. Goods belonging to *owner* have been sold without owner's consent by *transferrer* to *transferee* by a contract. (We can suppose that transferrer has stolen or hired the goods from owner and had it in possession at the time of the contract with transferee.) The normative problem is: Under what conditions is there an obligation (denoted O1) for transferrer to deliver the goods to owner? Under what conditions is there an obligation (denoted O2) for transferee to deliver the goods to owner?

We consider four systems and for all of them we assume that the stratum of grounds coincides with its reduct and similarly for the stratum of consequences, i.e. R_i coincides with $\leq_i$.

The example is a *cis*-application representing four normative systems with general norms where descriptive conditions imply normative conditions. For convenience, the conditions involved will be referred to in an abbreviated way. So, for example, condition P below ("Transferee has the goods in possession") refers to a complex condition $C(x_1, ..., x_n)$ fulfilled or not fulfilled by an n-tuple of individuals $\langle i_1, ..., i_n \rangle$ in a situation s. For details on conditions in the *cis* of the present example, the reader is referred to [Lindahl and Odelstad, 2003, pp. 86ff.].

The conditions dealt with in this example are the following (where $'$ signifies negation):

Grounds

P = Transferee has (= the transferrer has not) the goods in possession.

F = Transferee was in good faith at the time of the transfer.

R = the owner offers to pay ransom to transferee for the goods.

Normative consequences

O1 = Transferrer has an obligation to deliver the goods to owner.

O2 = Transferee has an obligation to deliver the goods to owner.

Verum and falsum

$\perp$ falsum

$\top$ verum

To simplify the example, we stipulate that it is assumed that the goods are either in the possession of transferrer or in the possession of transferee (no third possibility).

The example is intended to illustrate that, by means of Theorems 3.34 and 3.37, we get a test for whether a legal system is a joining-system, useful in situations of subtraction of norms from a system and addition of norms to a system.

We consider four systems, $\mathcal{S}_I, \mathcal{S}_{II}, \mathcal{S}_{III}, S_{IV}$, where

- $\mathcal{S}_I$ is a joining-system,

- $\mathcal{S}_{II}$, the result of subtraction from $\mathcal{S}_I$, is not a joining-system,
- $\mathcal{S}_{III}$, the result of a more comprehensive subtraction from $\mathcal{S}_I$, is a joining-system, and,
- S_{IV}, the result of an addition to $\mathcal{S}_{III}$, is a joining-system.

We make the following assumptions concerning the *Bqo*'s involved in the example:

1. The *Bqo*
$$\mathcal{B}_1 = \langle B_1, \wedge,', R_1\rangle, \text{ where } R_1 = \leq_1,$$
of grounds is the same for the systems $\mathcal{S}_I, \mathcal{S}_{II}, \mathcal{S}_{III}$; B_1 consists of the Boolean combinations of F and P.
(The *Bqo* of grounds in S_{IV} will be indicated later).
2. The *Bqo*
$$\mathcal{B}_2 = \langle B_2, \wedge,', R_2\rangle, \text{ where } R_2 = \leq_2,$$
of consequences is the same for all of $\mathcal{S}_I, \mathcal{S}_{II}, \mathcal{S}_{III}$, $\mathcal{S}_{IV}$; B_2 consists of the Boolean combinations of O1 and O2;

We introduce the following names for some of the norms in $\mathcal{S}_I - \mathcal{S}_{III}$:

$\mathbf{a} = \langle$F′∧P,O2$\rangle$
$\mathbf{b} = \langle$P,O1′$\rangle$
$\mathbf{c} = \langle$F∧P,O1′∧O2′$\rangle$
$\mathbf{d} = \langle$F′∨P′,O1∨O2$\rangle$
$\mathbf{e} = \langle$F∨P′,O2′$\rangle$
$\mathbf{f} = \langle$P′,O1$\rangle$
$\langle\bot,\bot\rangle$
$\langle\top,\top\rangle$

In *System* $\mathcal{S}_I$ (which is a *qo-corr* but, at this stage, not assumed to be a *Js*) the answer to the normative problem stated above depends on whether transferee has possession of the goods (denoted P) and whether transferee was in good faith at the time of the contract (denoted F). Let

$$K_I = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \langle\bot,\bot\rangle, \langle\top,\top\rangle\}$$

be the set of norms in $\mathcal{S}_I$ that are minimal with respect to $\trianglelefteq$. Figure 14 on page 601 shows the six minimal, non-degenerated norms and their interrelation in system $\mathcal{S}_I$: $\langle K_I, \precsim /K_I\rangle$ is a lattice, see Figure 15 on page 602. The

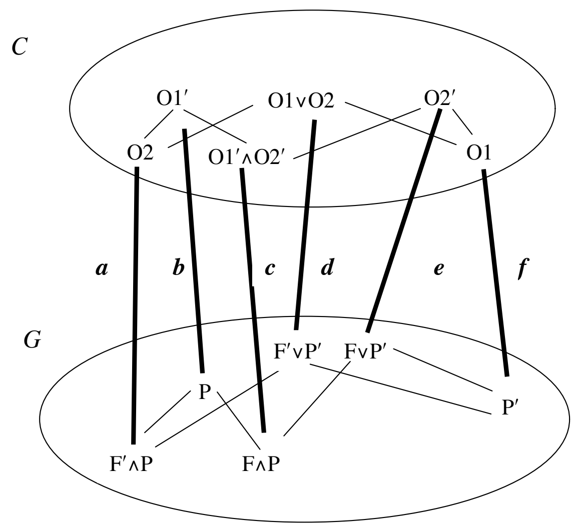


Figure 14

assumptions in Theorem 3.37 are satisfied. From Theorem 3.37 it follows that $\mathcal{S}_I = \langle \mathcal{B}_1, \mathcal{B}_2, \uparrow K_I \rangle$ is a *Bjs* and that $\min \uparrow K_I = K$.

We note that, for some $X \subseteq K_I$, $\langle \bot, \bot \rangle \in \mathrm{glb}_{\precsim} X$. Thus, for example, $\langle \bot, \bot \rangle \in \mathrm{glb}_{\precsim}\{\mathbf{a},\mathbf{c}\}$. Similarly, for some $X \subseteq K_I$, $\langle \top, \top \rangle \in \mathrm{lub}_{\precsim} X$. Thus,$\langle \top, \top \rangle \in \mathrm{lub}_{\precsim}\{\mathbf{b},\mathbf{d},\mathbf{e}\}$.

From the point of view of legal justice, *System* $\mathcal{S}_I$ may be thought to be unreasonable since it does not attach relevance to the possibility that owner can be willing to pay a ransom to transferee for getting the goods back. *System* $\mathcal{S}_{II}$ takes this consideration into account by elimination of some norms in the system. Suppose that the legislator in the set K_I of minimal joinings subtracts the minimal joining $\mathbf{c} = \langle$F∧P,O1′∧O2′$\rangle$, while $\mathbf{a}$, $\mathbf{b}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$, $\langle \bot, \bot \rangle$ and $\langle \top, \top \rangle$ are left.

System $\mathcal{S}_{II}$, where the set of minimal norms is

$$K_{II} = \{\mathbf{a}, \mathbf{b}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \langle \bot, \bot \rangle, \langle \top, \top \rangle\}$$

is a *qo-corr* but not a *Js*. Indeed, $\langle K_{II}, \precsim / K_{II} \rangle$ is a lattice, see Figure 16. Greatest lower bound of $\mathbf{b}$ and $\mathbf{e}$ in this lattice is $\langle \bot, \bot \rangle$, i.e. $\langle \bot, \bot \rangle \in \mathrm{glb}_{\precsim / K_{II}} \{\mathbf{b}, \mathbf{e}\}$. Note, however, that $\mathbf{c} \in \mathrm{glb}_{\precsim} \{\mathbf{b}, \mathbf{e}\}$. Hence, $\bot \in \pi_1 \left[\mathrm{glb}_{\precsim / K_{II}} \{\mathbf{b}, \mathbf{e}\} \right]$ but (F∧P)$\in \mathrm{glb}_{R_1} \pi_1 [\{\mathbf{b}, \mathbf{e}\}]$. And so, though

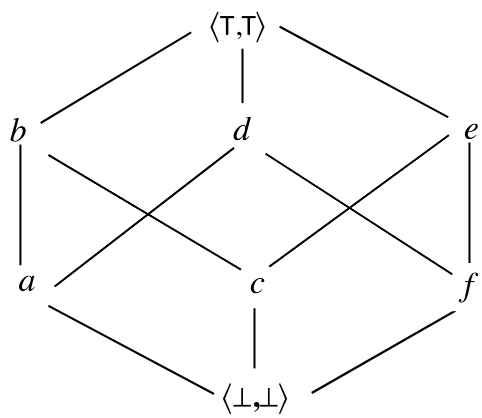


Figure 15

$\langle K_{II}, \precsim /K_{II}\rangle$ is a lattice (and complete since it is finite), it does not satisfy requirement (iii) in Theorem 3.34. Therefore, $\langle \mathcal{B}_1, \mathcal{B}_2, \uparrow K_{II}\rangle$ is not a *Js*.

If **c** is subtracted, in order to obtain a joining-system, the legislator has to subtract either **b** or **e,** or both, as well. Since elimination of **b** would seem unreasonable from a legal point of view, the appropriate choice would be to eliminate **e**. The resulting system will here be called *System* $\mathcal{S}_{III}$.

System $\mathcal{S}_{III}$ (which is a *qo-corr*, but, at this stage, is not assumed to be a joining-system) is such that

$$K_{III} = \{\mathbf{a}, \mathbf{b}, \mathbf{d}, \mathbf{f}, \langle \perp, \perp\rangle\, \mathbf{,}\, \langle \top, \top\rangle\}$$

See Figure 17.

$\langle K_{III}, \precsim_{K_{III}}\rangle$ is a lattice. See Figure 18. Moreover, the assumptions in Theorem 3.37 are satisfied. Hence, it follows that $S_{III} = \langle \mathcal{B}_1, \mathcal{B}_2, \uparrow K_{III}\rangle$ is a *Bjs*.

S_{III}, however, is legally unsatisfactory, since it is merely the result of subtraction, without positively stipulating anything about the relevance of owner's offering/not offering to pay ransom for the goods. The next system to be considered, therefore, is *System* S_{IV}, where "Ransom" is introduced. The *Bqo* of grounds in S_{IV} is

$$\mathcal{B}_3 = \langle B_3, \wedge, ', R_3\rangle \text{ with } R_3 = \leq_3;$$

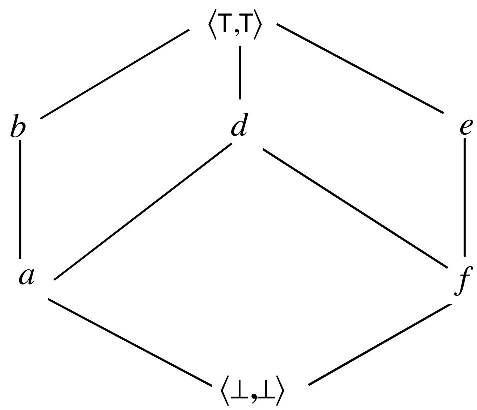


Figure 16

where B_3 consists of Boolean combinations of F, P and R. In S_{IV} the following norms are added:

$\langle \mathrm{P}\wedge\mathrm{R},\mathrm{O2}\rangle$. If transferee has the goods in possession and owner pays ransom for the good, then transferee has the obligation to deliver the good to owner.

$\langle \mathrm{F}\wedge\mathrm{P}\wedge\mathrm{R}',\mathrm{O2}'\rangle$. If transferee has the good in possession and fulfills the good faith condition, and owner does not pay ransom, then transferee has no obligation to deliver the good back to owner. These added norms however, are not minimal elements.

In $\mathcal{S}_{IV}$ (which is assumed to be a *qo-corr* but not a *Js*) the set of minimal norms is

$$K_{IV} = \{\mathbf{b}, \mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{i}, \mathbf{j}, \langle\bot,\bot\rangle, \langle\top,\top\rangle\}$$

where

$\mathbf{g} = \langle \mathrm{P}\wedge(\mathrm{F}'\vee\mathrm{R}),\mathrm{O2}\rangle$
$\mathbf{h} = \langle \mathrm{F}\wedge\mathrm{P}\wedge\mathrm{R}',\mathrm{O1}'\wedge\mathrm{O2}'\rangle$
$\mathbf{i} = \langle \mathrm{F}'\vee\mathrm{P}'\vee\mathrm{R},\mathrm{O1}\vee\mathrm{O2}\rangle$
$\mathbf{j} = \langle \mathrm{P}'\vee(\mathrm{F}\wedge\mathrm{R}'),\mathrm{O2}'\rangle$

We note that, of the non-degenerated minimal norms in the original system $\mathcal{S}_I$, only **b** and **f** remain unchanged in $\mathcal{S}_{IV}$, while, due to the relevance of ransom, **g**, **h**, **i**, **j** are new minimal norms in $\mathcal{S}_{IV}$.

The set of non-degenerated norms in K_{IV} and their interrelations is depicted in Figure 19. $\langle K_{IV}, \precsim / K_{IV}\rangle$ is a lattice, and hence complete, since it is finite. See Figure 20 on page 607. Moreover, the assumptions in Theorem

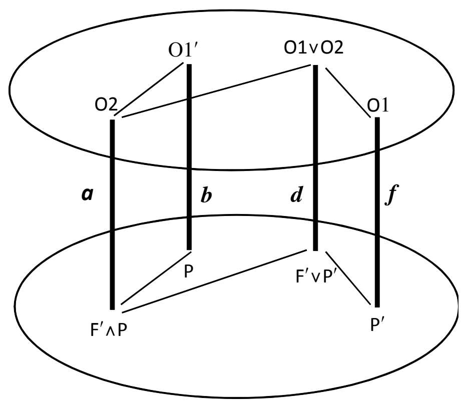


Figure 17

3.37 are satisfied and hence, it follows that $\langle \mathcal{B}_3, \mathcal{B}_2, \uparrow K_{IV} \rangle$ is a joining-system.[21] For further details on the example, cf. [Lindahl and Odelstad, 2003], developed within a slightly different framework (cf. Section 6.1 below).

4.4 The cis version of normative positions

The Kanger-Lindahl theory A natural approach to formulate normative concepts such as obligation and permission is to do so in terms of so-called *normative positions,* constructed by a combination of deontic logic and action logic. As is further developed in Marek Sergot's chapter "The theory of normative positions" of the present Handbook, the first version of the theory of normative positions, in its modern logical form, was developed by the Swedish logician Stig Kanger ([Kanger, 1957; Kanger, 1963]). Kanger's theory was inspired by the system of "fundamental jural relations" proposed by the American jurist W.N. Hohfeld in 1913. As realized by Kanger, standard deontic logic, with a deontic operator applied to sentences, is not adequate for expressing the Hohfeldian distinctions. The improvement proposed by Kanger was to combine a standard deontic operator *Shall* with an action operator *Do* (for "sees to it that") and to exploit the possibilities of external and internal negation of sentences where these operators are combined. Originally, Kanger's theory was conceived as a theory

[21]Basically, this was the system of Swedish legislation before 2003. That year, the law was changed so that, when the original owner has lost possession by theft, no ransom is required för getting the goods back.

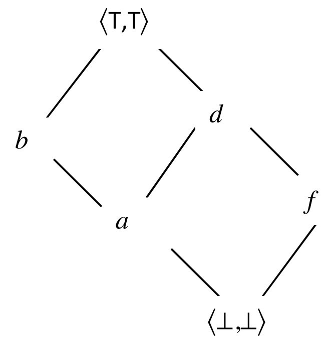


Figure 18

of *rights* (see [Lindahl, 1994]). As a theory of "legal" or "normative" positions, Kanger's theory was further developed by Lars Lindahl in [Lindahl, 1977]. Additional refinements of the so-called Kanger-Lindahl theory have been made by Andrew J.I. Jones and Marek Sergot ([Jones and Sergot, 1993; Jones and Sergot, 1996; Sergot, 1999; Sergot, 2001]). A special feature of the work of Jones and Sergot is that applications in computer science are in view.

A natural approach to the fine-grained structure of a *cis-Bjs* $\langle \mathcal{B}_1, \mathcal{B}_2, J \rangle$ where the stratum $\mathcal{B}_2$ is normative, is to formulate $\mathcal{B}_2$ in terms of an algebraic version of the Kanger-Lindahl theory of normative positions. (On this theory, see Sergot's chapter "The theory of normative positions" in the present Handbook.) The system of normative positions dealt with in what follows below is the system of *one-agent* types of normative position, in the sense of [Lindahl, 1977, ch. 3]. This system, chosen here since it is relatively simple, can easily be generalized to *n-agent* types, see Sergot's chapter and cf. Talja in [Talja, 1980].

To the Boolean connectives of negation, conjunction etc., are added the modal expressions "Shall" and "Do". If F is a state of affairs and x is an agent,[22] Shall F is to be read "It shall be the case that F" and $\mathrm{Do}(x, F)$ should be read "x sees to it that F". The expression May$F$ is an abbreviation for $\neg$Shall$\neg F$.

The basic idea in the Kanger-Lindahl theory is to exploit the possibilities of combining the deontic operator Shall with the action operator Do. One example is Shall $\mathrm{Do}(x, F)$ which means that it shall be that x sees to it that

[22] A state of affairs in Kanger's sense might be, for example, that Mr. Smith gets back the money lent by him to Mr. Black, or that Mr. Smith walks outside Mr. Black's shop.

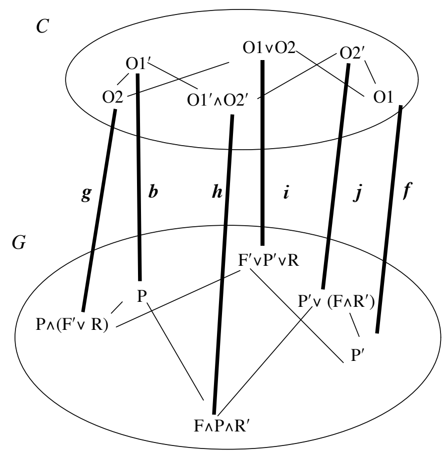


Figure 19

F; another is $\neg$ Shall Do$(y, \neg F)$ which means that it is not the case that it shall be that y sees to it that not F.

The logical postulates for Shall and Do assumed in the construction of one-agent types are as follows (cf. [Lindahl, 1977, p. 68]):

Rules for Do

RI. If $\vdash (A \longleftrightarrow B)$, then $\vdash (\text{Do}(s, A) \longleftrightarrow \text{Do}(s, B))$.
A1. $\text{Do}(s, A) \to A$.

Rules for Shall

RII. If $\vdash A$, then $\vdash \text{Shall}A$.
A2. Shall $(A \to B) \to (\text{Shall}A \to \text{Shall}B)$.
A3. $\text{Shall}A \to \neg\text{Shall}\neg A$.

The systems of normative positions can serve as a tools for describing the normative positions of different agents $x, y, z...$ with regard to states of affairs $F, G, H, ...$ For example, if x is the Swedish Government and F is the state of affairs that a paper on normative positions by Sergot is published in Sweden, the position, according to Swedish law, of x with regard to F can be described by Shall$(\neg\text{Do}(x, F)$ & $\neg\text{Do}(x, \neg F))$, expressing that the Government is not allowed either to bring about or prevent the publication.

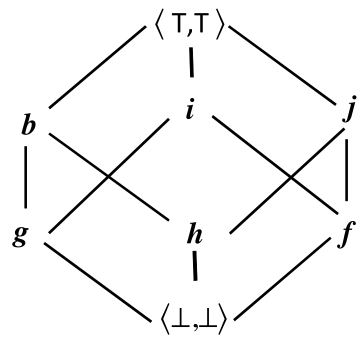


Figure 20

If x is an agent and F is a state of affairs, the seven one-agent types of position are as follows (see [Lindahl, 1977, p. 92]), where $\mathrm{Pass}(x, F)$ is an abbreviation for $\neg\mathrm{Do}(x, F)$ & $\neg\mathrm{Do}(x, \neg F)$:

$T_1(x, F)$: $\mathrm{MayDo}(x, F)$ & $\mathrm{MayPass}(x, F)$ & $\mathrm{MayDo}(x, \neg F)$.
$T_2(x, F)$: $\mathrm{MayDo}(x, F)$ & $\mathrm{MayPass}(x, F)$ & $\neg\mathrm{MayDo}(x, \neg F)$.
$T_3(x, F)$: $\mathrm{MayDo}(x, F)$ & $\neg\mathrm{MayPass}(x, F)$ & $\mathrm{MayDo}(x, \neg F)$.
$T_4(x, F)$: $\neg\mathrm{MayDo}(x, F)$ & $\mathrm{MayPass}(x, F)$ & $\mathrm{MayDo}(x, \neg F)$.
$T_5(x, F)$: $\mathrm{MayDo}(x, F)$ & $\neg\mathrm{MayPass}(x, F)$ & $\neg\mathrm{MayDo}(x, \neg F)$.
$T_6(x, F)$: $\neg\mathrm{MayDo}(x, F)$ & $\mathrm{MayPass}(x, F)$ & $\neg\mathrm{MayDo}(x, \neg F)$.
$T_7(x, F)$: $\neg\mathrm{MayDo}(x, F)$ & $\neg\mathrm{MayPass}(x, F)$ & $\mathrm{MayDo}(x, \neg F)$.

The numbering of the T_i conforms to the numbering of the corresponding one-agent types of normative position in [Lindahl, 1977]. The numbering suits the representation of the types in a Hasse diagram, exhibiting how the types are partially ordered by the relation "less free than" (see [Lindahl 1977, pp. 105 ff]).

The simplest way to combine the TJS approach with an algebraic version of the theory of one-agent normative positions is to transform the one-agent formulas $T_1(x, F), ..., T_7(x, F)$ into seven *conditions* $T_1q, ..., T_7q$. Thus T_i, when occurring in T_iq, is an operator on conditions, and the result is a normative condition, defined in terms of one-agent type T_i. A set $\{T_1q, ..., T_7q\}$ of seven normative conditions is obtained, and Boolean compounds of these seven conditions are formed by $\wedge, ', \vee$.

Next we construct a *normative position cis.* Let $\mathcal{B} = \langle B, \wedge, ', R\rangle$ be a *cis-Bqo* with a domain B of descriptive conditions $q_1, q_2, ...$. Furthermore, let

$$T_{\mathcal{B}} = \{T_i q \mid q \in B - \{\bot, \top\}, 1 \leq i \leq 7\},$$

i.e., $T_{\mathcal{B}}$ is the set of all normative positions with regard to the descriptive conditions in B. Next, let $T^*_{\mathcal{B}}$ be the closure of $T_{\mathcal{B}}$ under $\wedge,'$. Then $\mathcal{T} = \langle T^*_{\mathcal{B}}, \wedge,' \rangle$ is a Boolean algebra, called a *Boolean normative position algebra.*

Finally, from $\mathcal{T}$ we construct a *cis-Bqo* $\langle T^*_{\mathcal{B}}, \wedge,', R\rangle$, called a *normative position cis.* Such as *cis* is to fulfil the requirements of deontic logic and action logic described in the theory of one-agent normative positions. These requirements are incorporated in the following definition.

Definition 4.5 *A* cis $\langle T^*_{\mathcal{B}}, \wedge,', R\rangle$ *is a* normative position cis *with regard to* $\mathcal{B}$ *if for any* $q, r \in \mathcal{B}$ *it holds that*
(1) *if* $i \neq j$, *then* $T_i q \wedge T_j q \; R \perp$ (*for* $i, j \in \{1, ..., 7\}$),
(2) $\top \; R \; (T_1 q \vee ... \vee T_7 q)$,
(3) $T_1 q \, Q \, T_1 q'$, $T_3 q \, Q \, T_3 q'$, $T_6 q \, Q \, T_6 q'$, $T_2 q \, Q \, T_4 q'$, $T_5 q \, Q \, T_7 q'$,
(4) *if* $q \, Q \, r$, *then* $T_i q \; Q \; T_i r$,
(5) *if* $i = 1, 3, 4, 7$, *then* $T_i \top \, Q \perp$, *and,*
(6) *if* $i = 1, 2, 3, 5$, *then* $T_i \bot \, Q \perp$.

Requirements (1)-(4) in the definition express restrictions on the relation R in a normative position algebra and correspond to three features of one agent types in the Kanger-Lindahl theory. Thus requirement (1) expresses that $T_1 q, ..., T_7 q$ are mutually incompatible, (2) that they are jointly exhaustive, and (3) that T_1, T_3, T_6 are neutral, while T_4 is the converse of T_2 and T_7 the converse of T_5. Requirements (4)-(6), finally, follow from the logic of Shall and Do, where (4) corresponds to the "extensionality" feature for combinations of operators Shall and Do in the Kanger-Lindahl theory, and (5) and (6) follow from the theorem $\neg \text{MayDo}(x, \bot)$. (See [Lindahl and Odelstad, 2004, sect. 1.2, 4 and 6] for details.)

Liberty conditions For seeing more clearly what various conditions in a normative position *cis* amount to in deontic terms, the notion of *liberty conditions* can be introduced (cf. Lindahl 1977, pp. 106 ff.). This device is available since each normative position condition equals a Boolean compound of liberty conditions.

There are three liberty operators L_1, L_2 and L_3. These can be called action permissibility, passivity permissibility and counter-action permissibility, respectively. In terms of May and Do we can read non-negated liberty conditions as follows.

Action permissibility: L_1
$L_1 q(x_1, ..., x_\nu, x_{\nu+1})$ iff May $\text{Do}(x_{\nu+1}, q(x_1, ..., x_\nu))$

Passivity permissibility: L_2
$L_2q(x_1, ..., x_\nu, x_{\nu+1})$ iff May Pass$(x_{\nu+1}, q(x_1, ..., x_\nu))$

Counter-action permissibility: L_3
$L_3q(x_1, ..., x_\nu, x_{\nu+1})$ iff May Do$(x_{\nu+1}, q(x_1, ..., x_\nu)')$

Liberty conditions L_1, L_2, L_3 can be defined in terms of disjunctions of basic *np*-conditions.

Definition 4.6 *L_1, L_2, L_3 are operators on conditions such that, if q is a condition:*
(1) L_1q is defined as: $T_1q \vee T_2q \vee T_3q \vee T_5q$.
(2) L_2q is defined as: $T_1q \vee T_2q \vee T_4q \vee T_6q$.
(3) L_3q is defined as: $T_1q \vee T_3q \vee T_4q \vee T_7q$.

Accordingly, it holds that (where $'$ signifies negation),
$T_1q \ Q \ L_1q \wedge L_2q \wedge L_3q$,
$T_2q \ Q \ L_1q \wedge L_2q \wedge (L_3q)'$,
$T_3q \ Q \ L_1q \wedge (L_2q)' \wedge L_3q$,
$T_4q \ Q \ (L_1q)' \wedge L_2q \wedge L_3q$,
$T_5q \ Q \ L_1q \wedge (L_2q)' \wedge (L_3q)'$,
$T_6q \ Q \ (L_1q)' \wedge L_2q \wedge (L_3q)'$,
$T_7q \ Q \ (L_1q)' \wedge (L_2q)' \wedge L_3q$.

Accordingly, if L_{iq} is denoted by 1 and $(L_{iq})'$ by 0, the basic *np*-conditions can be represented by the semi-lattice in Figure 21 (cf. [Lindahl, 1977, p. 105] and [Talja, 1980]).

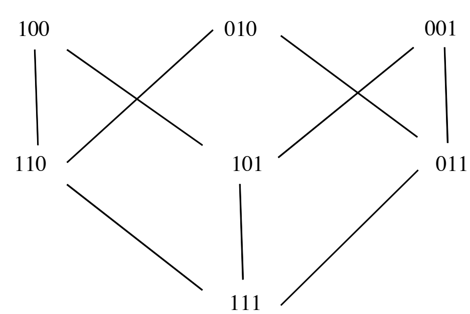


Figure 21

4.4.1 An example: ownership to an estate

Suppose we represent a normative system by a *cis* model of a joining-system with two strata one of which is a descriptive *cis*, and the other is a *normative position-cis*. We illustrate this representation by a simple example concerning the normative position of owners of real property in a legal system $\mathcal{S}$. We consider a *cis* model of a Boolean joining-system $\langle \mathcal{B}_1, \mathcal{B}_2, J \rangle$ where $\mathcal{B}_1 = \langle B_1, \wedge, ', R_1 \rangle$ is descriptive, while $\mathcal{B}_2 = \langle B_2, \wedge, ', R_2 \rangle$ is a *normative position-cis*.

The two strata considered

The descriptive stratum $\mathcal{B}_1$.

We assume that conditions a_1 and b_1, appearing in the descriptive lower stratum $\mathcal{B}_1$ are as follows:

a_1: Being the owner of an estate E.[23]

b_1: Being the owner of an estate adjacent to estate E.

We furthermore assume that $\mathcal{B}_1$ is as depicted in the following diagram (where $\alpha \bowtie \beta$ is an abbreviation for $(\alpha \wedge \beta) \vee (\alpha' \wedge \beta')$ and where lines representing R_1 (implication) are omitted as being evident):

$$
\begin{array}{lcccccccc}
\mathcal{B}_1 & & & & \top & & & & \\
 & & a_1 \vee b_1 & a_1 \vee b_1' & & a_1' \vee b_1 & a_1' \vee b_1' & & \\
 & a_1 & b_1 & a_1 \bowtie b_1 & & a_1 \bowtie b_1' & b_1' & a_1' & \\
 & & a_1 \wedge b_1 & a_1 \wedge b_1' & & a_1' \wedge b_1 & a_1' \wedge b_1' & & \\
 & & & & \bot & & & &
\end{array}
$$

We note that $\mathcal{B}_1$ coincides with its reduct $\langle B_1, \wedge, ' \rangle$ and that, therefore, in $\mathcal{B}_1$, R_1 coincides with $\leq_1$. As appears from the diagram, it is assumed that conditions $a_1 \wedge b_1, a_1 \wedge b_1', a_1' \wedge b_1, a_1' \wedge b_1'$ are atoms in $\mathcal{B}_1$.

[23]Letter E is to be regarded as a parameter, in the sense of a a quantity which is constant in a particular case considered, but which varies in different cases.

The normative stratum $\mathcal{B}_2$
Let conditions $q_1, ..., q_4$ be as follows:

q_1: Main building of estate E being painted white,
q_2: Main building on estate adjacent to E being painted white,
q_3: Cows of estate E entering land of adjacent estate,
q_4: Erecting a fence, going around estate E and adjacent estate.

Let $\mathcal{B} = \langle B, \wedge,' R\rangle$ be a *cis* such that the descriptive conditions q_1, q_2, q_3, q_4 are among the elements of its domain. Furthermore, as in Section 4.4, let $T_{\mathcal{B}} = \{T_i q \mid q \in B - \{\bot, \top\}, 1 \leq i \leq 7\}$, let $T^*_{\mathcal{B}}$ be the closure of $T_{\mathcal{B}}$ under $\wedge,'$ and let $\mathcal{T} = \langle T^*_{\mathcal{B}}, \wedge,' \rangle$ be a Boolean normative position algebra with regard to $\mathcal{B}$. Finally, let $\mathcal{B}_2 = \langle T^*_{\mathcal{B}}, \wedge,', R_2\rangle$ be a normative position *cis* with regard to $\mathcal{B}$ (see above definition 4.5). Since $\mathcal{T}$ is the reduct of $\mathcal{B}_2$, the Boolean relation $\leq_T$ of $\mathcal{T}$ is a subset of the relation R_2 of $\mathcal{B}_2$.

Joining assumptions
We assume that in the Boolean joining-system $\langle \mathcal{B}_1, \mathcal{B}_2, J\rangle$, when referring to non-degenerated joinings, the following holds:

(i) $(a_1 \wedge b_1)\ J\ (T_1q_1 \wedge T_1q_2 \wedge T_1q_3 \wedge T_1q_4)$,

(ii) $(a_1 \wedge b_1')\ J\ (T_1q_1 \wedge T_6q_2 \wedge T_7q_3 \wedge T_4q_4)$,

(iii) $(a_1' \wedge b_1)\ J\ (T_6q_1 \wedge T_1q_2 \wedge T_4q_3 \wedge T_4q_4)$,

(iv) $(a_1' \wedge b_1')\ J\ (T_6q_1 \wedge T_6q_2 \wedge T_6q_3 \wedge T_6q_4)$.

Given the intended interpretation of conditions T_iq_j in terms of Shall, May and Do, the joinings (i)-(iv) are plausible for a legal system. This can be seen by inspection of the different grounds and consequences correlated. For this purpose, the notion of liberty conditions is useful (on liberty conditions, see above Section 4.4). To exemplify, $a_1 \wedge b_1$ means being the owner of both estate E and adjacent estate. This condition is a ground for $T_1q_1\wedge T_1q_2\wedge T_1q_3\wedge T_1q_4$, which is the normative position-condition denoting full freedom (operator T_1) with regard to all of $q_1, ..., q_4$ (painting the two buildings, letting the cows move around, erecting a surrounding fence). In contrast, $a_1 \wedge b_1'$ means owning estate E but not adjacent estate. This condition is ground for $T_1q_1 \wedge T_6q_2 \wedge T_7q_3 \wedge T_4q_4$.This condition denotes full freedom regarding the painting of building on estate E, no freedom to bring about or prevent painting of building on adjacent estate, obligation to see to it that cows from estate E do not enter land of adjacent estate, and, finally, freedom to prevent erection of the fence surrounding the estates and freedom to be passive about the matter, but no freedom to bring about the fence's being erected.

For further development of the example, see [Lindahl and Odelstad, 2004, sect. 6].

5 Intervenients for Boolean joining-systems

5.1 Introductory remarks on intervenients in Bjs'

In the present main section (Section 5) we will investigate the structure of a stratum $\langle B_2, R_2 \rangle$ with intervenients, between one stratum $\langle B_1, R_1 \rangle$ of grounds and one stratum $\langle B_3, R_3 \rangle$ of consequences. In the present first subsection (Section 5.1), we introduce some notation and some basic results, in particular as regards Boolean operations on intervenients. Since these remarks have been dealt with extensively in [Lindahl and Odelstad, 2011], the general remarks are kept brief, and the reader is referred to [Lindahl and Odelstad, 2011] for proofs and further details.

One possible use of intervenients, not dealt with in the present chapter, is for characterizing a Boolean joining-system. Intervenients from B_1 to B_3 can be used for defining or characterizing the Boolean joining-system $\langle \mathcal{B}_1, \mathcal{B}_3, J_{1,3} \rangle$. Cf. [Lindahl and Odelstad, 2008a, sect. 2.3.5 and 4], on *gic*-systems, proto-intervenients and the methodology of intermediate concepts.

After these remarks, attention will be paid in particular to *cis* applications regarding some important issues. In particular, networks of strata with intervenients, organic wholes of intervenients and narrowing of intervenients will be dealt with.

In Section 3.8, the notion of an intervenient was defined with respect to simple *Js*-triples presupposing that the joinings of the strata are disjunct sets. This presupposition is not appropriate when it comes to intervenients in systems of *Bjs*'s, which can be seen in the following way. Suppose that $\mathcal{S}_1 = \langle \mathcal{B}_1, \mathcal{B}_2, J_{1,2} \rangle$, $\mathcal{S}_2 = \langle \mathcal{B}_2, \mathcal{B}_3, J_{2,3} \rangle$ and $\mathcal{S}_3 = \langle \mathcal{B}_1, \mathcal{B}_3, J_{1,3} \rangle$, where $\mathcal{B}_i = \langle B_i, \wedge, ', R_i \rangle$, are *Bjs*'s and that $B_i \cap B_j = \{\bot, \top\}$ if $i \neq j$, $1 \leq i, j \leq 3$. Then it can be the case that for some $a_2 \in B_2$, $\bot$ is the weakest ground of a_2 or $\top$ is the strongest consequence of a_2. In either case, a_2 is not a proper intervenient since $\langle \bot, a_2 \rangle$ and $\langle a_2, \top \rangle$ are degenerated joinings (cf. Section 4.1.2). We say that a_2 is a non-degenerated intervenient if a_2 is an intervenient and $a_2 \curvearrowright \langle a_1, a_3 \rangle$, where $\langle a_1, a_3 \rangle$ is a non-degenerated joining.

Definition 5.1 *Suppose that* $\mathcal{S}_1 = \langle \mathcal{B}_1, \mathcal{B}_2, J_{1,2} \rangle$*,* $\mathcal{S}_2 = \langle \mathcal{B}_2, \mathcal{B}_3, J_{2,3} \rangle$ *and* $\mathcal{S}_3 = \langle \mathcal{B}_1, \mathcal{B}_3, J_{1,3} \rangle$ *are joining-systems where* $\mathcal{B}_i = \langle B_i, \wedge, ', R_i \rangle$ *are complete and* $B_i \cap B_j = \{\bot, \top\}$ *for* $i \neq j$*,* $1 \leq i, j \leq 3$*. If* $J_{1,3} \supseteq J_{1,2}|J_{2,3}$ *we say that* $\langle \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \rangle$ *is a* Bjs-triple.

(Concerning completeness, see Section 4.1.1.)

Definition 5.2 *In a* Bjs*-triple* $\langle \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \rangle$*, the element* $a_2 \in B_2$*, is a non-degenerated* intervenient from $\mathcal{B}_1$ to $\mathcal{B}_3$ corresponding to *the joining* $\langle a_1, a_3 \rangle \in J_{1,3}$*, denoted* $a_2 \curvearrowright \langle a_1, a_3 \rangle$ *, if* a_1 *is a non-degenerated weakest ground of* a_2 *in* $\mathcal{S}_1$ *and* a_3 *is a non-degenerated strongest consequence of* a_3 *in* $\mathcal{S}_2$*.*

Suppose that $\Phi = \langle \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \rangle$ is a *Bjs*-triple. Note that if $a_2 \in B_2$ is an intervenient in Φ from $\mathcal{B}_1$ to $\mathcal{B}_3$ then there is $a_1 \in B_1$ and $a_3 \in B_3$ such that a_2 is situated between B_1 and B_3 in $\mathcal{S}$ in the sense that $\langle a_1, a_2 \rangle \in J_{1,2}$, $\langle a_2, a_3 \rangle \in J_{2,3}$ and $\langle a_1, a_3 \rangle \in J_{1,3}$. Now, let us look at the converse of this statement. Suppose that $\langle a_1, a_2 \rangle \in J_{1,2}$, $\langle a_2, a_3 \rangle \in J_{2,3}$ and $\langle a_1, a_3 \rangle \in J_{1,3}$. Then, if a_1 is not similar to falsum and a_3 not similar to verum, then a_2 is an intervenient from $\mathcal{B}_1$ to $\mathcal{B}_3$. However, it is important to notice that, even though a_2 is an intervenient from $\mathcal{B}_1$ to $\mathcal{B}_3$ in Φ, it is not guaranteed that $a_2 \curvearrowright \langle a_1, a_3 \rangle$, i.e., that a_2 corresponds to $\langle a_1, a_3 \rangle$. But if $\langle a_1, a_2 \rangle \in \min J_{1,2}$, and $\langle a_2, a_3 \rangle \in \min J_{2,3}$, this holds. Note also that if $\langle a_1, a_3 \rangle \in \min J_{1,3}$ then there is $b_2 \in B_2$ such that b_2 is an intervenient in Φ from $\mathcal{B}_1$ to $\mathcal{B}_3$ and $b_2 \curvearrowright \langle a_1, a_3 \rangle$. (See [Lindahl and Odelstad, 2004, sect. 4] for details.)

5.1.1 Conjunction, disjunction and negation of intervenients

If we apply the Boolean operations conjunction, disjunction and negation on intervenients, will the result be intervenients as well? Which is the relationship between the conjunction of the weakest grounds of two intervenients and the weakest ground of their conjunction, and similarly for disjunction and negation? The same question arises with regard to strongest consequences. We will here consider conjunction and disjunction of pairs of intervenients. Of special interest is Boolean operations in connection with minimality.

Conjunction and disjunction of intervenients

In a *Bjs-triple* $\Phi = \langle \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \rangle$, we let $\mathrm{Iv}\,(B_2, B_1, B_3)$ denote the set of elements in B_2 which are intervenients from B_1 to B_3 in Φ. We state some results presented in [Lindahl and Odelstad, 2011, sect. 4.2].

The following theorem states a necessary and sufficient condition for a conjunction of intervenients being an intervenient, and similarly for a disjunction of intervenients.

Theorem 5.3 *Suppose that* $\mathcal{B}_1$ *and* $\mathcal{B}_3$ *are complete and that* $a_2 \curvearrowright \langle a_1, a_3 \rangle$ *and* $b_2 \curvearrowright \langle b_1, b_3 \rangle$*. Then*

1. $\perp P_1(a_1 \wedge b_1)$ *iff* $(a_2 \wedge b_2) \in \mathrm{Iv}\,(B_2, B_1, B_3)$ *, and*

2. $(a_3 \vee b_3)\, P_3 \top$ *iff* $(a_2 \vee b_2) \in \mathrm{Iv}\,(B_2, B_1, B_3)$.

The following theorem states the relationships between the Boolean operations on intervenients and the corresponding operations on grounds and consequences, respectively. These relationships are important for the discussion of organic wholes of intervenients in the Section 5.2.1.

Theorem 5.4 *Suppose that* $\mathcal{B}_1$ *and* $\mathcal{B}_3$ *are complete and that* $a_2 \curvearrowright \langle a_1, a_3 \rangle$, $b_2 \curvearrowright \langle b_1, b_3 \rangle$. *Then,*

1. *If* $(a_2 \wedge b_2) \in \mathrm{Iv}\,(B_2, B_1, B_3)$ *then there is* $c_3 \in B_3$ *such that* $a_2 \wedge b_2 \curvearrowright \langle a_1 \wedge b_1, c_3 \rangle$.
2. *If* $(a_2 \vee b_2) \in \mathrm{Iv}\,(B_2, B_1, B_3)$ *then there is* $c_1 \in B_1$ *such that* $a_2 \vee b_2 \curvearrowright \langle c_1, a_3 \vee b_3 \rangle$.

The following theorems connect Boolean operations of intervenients to minimality.

Theorem 5.5 *Suppose that* $a_2 \curvearrowright \langle a_1, a_3 \rangle \in \min J_{1,3}$ *and* $b_2 \curvearrowright \langle b_1, b_3 \rangle \in \min J_{1,3}$ *and not* $a_1 \wedge b_1 R_1 \bot$ *and not* $\top R_3 a_3 \vee b_3$. *Then the following holds:*

1. *If* $\langle a_1 \wedge b_1, a_3 \wedge b_3 \rangle \in \min J_{1,3}$, *then* $a_2 \wedge b_2 \curvearrowright \langle a_1 \wedge b_1, a_3 \wedge b_3 \rangle$.
2. *If* $\langle a_1 \vee b_1, a_3 \vee b_3 \rangle \in \min J_{1,3}$, *then* $a_2 \vee b_2 \curvearrowright \langle a_1 \vee b_1, a_3 \vee b_3 \rangle$.

Theorem 5.6 *Suppose that* $a_2 \curvearrowright \langle a_1, a_3 \rangle \in \min J_{1,3}$ *and* $b_2 \curvearrowright \langle b_1, b_3 \rangle \in \min J_{1,3}$ *and, furthermore, not* $a_1 \wedge b_1 R_1 \bot$ *and not* $\top R_3 a_3 \vee b_3$. *Then there are* $c_2, d_2 \in B_2$, $c_3 \in B_3$ *and* $d_1 \in B_1$ *such that*

1. $c_2 \curvearrowright \langle a_1 \wedge b_1, c_3 \rangle \in \min J_{1,3}$, *where* $c_3 R_3 (a_3 \wedge b_3)$, *and*
2. $d_2 \curvearrowright \langle d_1, a_3 \vee b_3 \rangle \in \min J_{1,3}$, *where* $(a_1 \vee b_1) R_1 d_1$.

Negations of intervenients

Negations of intervenients is an interesting subject. We will here give an overview. (For details and proofs, see [Lindahl and Odelstad, 2008a]). Suppose that a_2 is an intervenient from $\mathcal{B}_1$ to $\mathcal{B}_3$ corresponding to the joining $\langle a_1, a_3 \rangle \in J_{1,3}$ in the *Bjs*-triple $\Psi = \langle \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \rangle$. Then there are two possibilities with regard to the negation a_2' of a_2:

1. a_2' is an intervenient from $\mathcal{B}_1$ *to* $\mathcal{B}_3$ in the *Bjs*-triple Ψ.
2. a_2' is not an intervenient from $\mathcal{B}_1$ *to* $\mathcal{B}_3$ in the *Bjs*-triple Ψ.

If a_2' is *not* an intervenient we can distinguish between three possibilities:

(i) a_2' has a non-degenerated weakest ground in B_1 but no non-degenerated strongest consequence in B_3.

(ii) a_2' has no non-degenerated weakest ground in B_1 but a non-degenerated strongest consequence in B_3.

(iii) a_2' has neither a non-degenerated weakest ground in B_1 nor a non-degenerated strongest consequence in B_3.

If a_2' *is* an intervenient it is important to note the relation between the joining corresponding to a_2 and to a_2'. Suppose that $a_2 \curvearrowright \langle a_1, a_3 \rangle$ and $a_2' \curvearrowright \langle b_1, b_3 \rangle$. Then:

(I) $\langle a_1', a_3' \rangle \trianglelefteq \langle b_1, b_3 \rangle$.

(II) If $\langle a_1, a_3 \rangle \in \min J_{1,3}$, then $\langle a_1', a_3' \rangle \backsimeq \langle b_1, b_3 \rangle$.

(III) If $\langle a_1', a_3' \rangle , \langle b_1', b_3' \rangle \in J_{1,3}$, then $\langle a_1', a_3' \rangle \backsimeq \langle b_1, b_3 \rangle$.

Note that if a_2' is an intervenient this constitutes a restriction on the possibility of narrowing a_2 (see Section 5.2.2 below), since a narrowing of a_2 implies a widening of $\langle a_1', a_3' \rangle$, and (I) above gives a restriction of how wide $\langle a_1', a_3' \rangle$ can be. If $a_2 \curvearrowright \langle a_1, a_3 \rangle$ and $\langle a_1, a_3 \rangle \in \min J_{1,3}$ and a_2' is an intervenient, then a_2 cannot be narrowed. The same holds if $a_2 \curvearrowright \langle a_1, a_3 \rangle$, $a_2' \curvearrowright \langle b_1, b_3 \rangle$ and $\langle a_1', a_3' \rangle , \langle b_1', b_3' \rangle \in J$. The subject of negations of intervenients is important in connection with open intermediaries (see Section 5.2.2 below).

5.2 cis' with intervenients

As appears from the foregoing, in TJS for intervenients, "intervenient" is a technical notion defined at the abstract algebraic level. The notion is intended as a tool for analyzing different kinds of what, informally, is called "intermediaries" and the aim is to provide tools for analyzing intermediaries as they appear in law, language, morals, and so on. For this reason *cis*' with intervenients is an important part of the chapter.

In the present Section 5.2, we assume that intervenients referred in the text are non-degenerated intervenients (see Definition 5.2).

5.2.1 Organic wholes

Attention should be drawn to the possible occurrence in normative systems of a phenomenon analogous to what G.E. Moore in *Principia Ethica* (first published in 1903) called an "organic unity" or "organic whole". Characteristic of an organic unity, according to Moore, is "that the value of such a whole bears no regular proportion to the sum of the values of its parts"[Moore, 1971, p. 27]. Using another terminology, the phenomenon can be called "synergy". In a context of norms, and within our algebraic framework of Boolean joining-systems, the idea of organic wholes refers to the normative impact of a Boolean compound of conditions rather than to "values" in Moore's sense. In the present section, this theme is dealt with as regards the normative impact of conjunction and disjunction of intervenients.

Definition 5.7 *Let* $a_2 \curvearrowright \langle a_1, a_3 \rangle$, $b_2 \curvearrowright \langle b_1, b_3 \rangle$*, and* $(a_2 \wedge b_2), (a_2 \vee b_2) \in \mathrm{Iv}(B_2, B_1, B_3)$.
(i) If there is $c_3 \in B_3$ *such that* $a_2 \wedge b_2 J_{2,3} c_3$ *and* $c_3 P_3 a_3 \wedge b_3$*, we say that* $a_2 \wedge b_2$ *is* a conjunctive organic whole *of* a_2 *and* b_2*,*
(ii) If there is $c_1 \in B_1$ *such that* $c_1 J_{1,2} a_2 \vee b_2$ *and* $a_1 \vee b_1 P_1 c_1$*, we say that* $a_2 \vee b_2$ *is* a disjunctive organic whole *of* a_2 *and* b_2*.*

Note that a disjunctive organic whole is constructed as the dual of a conjunctive organic whole.

A *cis* example of a conjunctive organic whole is a follows (cf. [Lindahl and Odelstad, 2003, sect. 5.1, p. 101]):

We imagine an athletic competition, where there are two events, running and high jumping. We consider three *Bqo*'s where $\mathcal{B}_1$ (with $a_1, b_1, ...$) concerns competition *results* in the two events, where $\mathcal{B}_2$ (with $a_2, b_2, ...$) concerns winner's *titles*, and where $\mathcal{B}_3$ (with $a_3, b_3, c_3, ...$) concerns rights to competition *prizes*.

a_1 is to be the fastest runner, b_1 is to jump the highest,

a_2 is to be "master of running", b_2 is to be "master of jumping", $a_2 \wedge b_2$ is to be "twofold master".

a_3 is to have the right of the running prize, b_3 is to have the right of the jumping prize, $c_3 = a_3 \wedge b_3 \wedge d_3$ is to have the right of the excellence prize, namely (a_3) the right of the running prize, and (b_3) the right of the jumping prize, and, in addition, (d_3) the right of a special bonus prize for the twofold master. The example is illustrated in Figure 22.

In the example we have: $a_2 \curvearrowright \langle a_1, a_3 \rangle$, $b_2 \curvearrowright \langle b_1, b_3 \rangle$, $a_2 \wedge b_2 \curvearrowright \langle a_1 \wedge b_1, c_3 \rangle$, where $c_3 P_3 (a_3 \wedge b_3)$. Since we have $c_3 P_3 (a_3 \wedge b_3)$, it holds in the *Bjs-triple* $\langle \langle \mathcal{B}_1, \mathcal{B}_2, J_{1,2} \rangle, \langle \mathcal{B}_2, \mathcal{B}_3, J_{2,3} \rangle, \langle \mathcal{B}_1, \mathcal{B}_3, J_{1,3} \rangle \rangle$ that the intervenient $a_2 \wedge b_2$ is an organic whole in relation to $\mathcal{B}_3$. In other words: $a_2 \wedge b_2$ is an organic whole since the consequence $c_3 = a_3 \wedge b_3 \wedge d_3$ of the intervenient $a_2 \wedge b_2$ is "stronger" (P_3) than the "sum" $a_3 \wedge b_3$ of the consequence a_3 of a_2 and the consequence b_3 of b_2.

A subset of the minimal joinings from B_2 to B_3 is depicted by the thick lines in Figure 22.

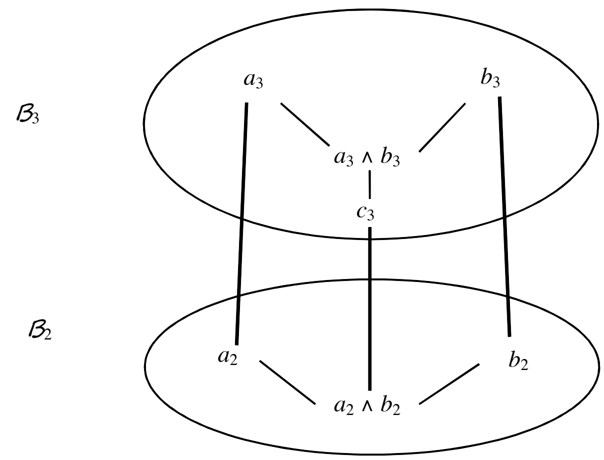


Figure 22

We observe that, in the sense of Theorem 3.34,

$$\mathrm{glb}_{R_2}\,\pi_1\{\langle a_2, a_3\rangle, \langle b_2, b_3\rangle\} = \mathrm{glb}_{R_2}\{a_2, b_2\} = \{a_2 \wedge b_2\} = \pi_1[\mathrm{glb}_{\precsim/\min J}\{\langle a_2, a_3\rangle, \langle b_2, b_3\rangle\}].$$

For a legal example concerning citizenship, see [Lindahl and Odelstad, 2003, sect. 5.1].

5.2.2 Open concepts and the narrowing of intervenients

Ground-narrowing We recall the issue of open legal concepts and the example of "relationship similar to being married" (Section 1.7.5 above). Let $\Psi = \langle \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3\rangle$ be a *Bjs-triple* with

$$\mathcal{S}_1 = \langle \mathcal{B}_1, \mathcal{B}_2, J_{1,2}\rangle,\ \mathcal{S}_2 = \langle \mathcal{B}_2, \mathcal{B}_3, J_{2,3}\rangle,\ \mathcal{S}_3 = \langle \mathcal{B}_1, \mathcal{B}_3, J_{1,3}\rangle.$$

Condition $a_2 \in B_2$ (where B_2 is the domain of stratum $\mathcal{B}_2$) is the condition of having a relationship similar to being married. The grounds for a_2 are among the elements of the domain B_1 of stratum $\mathcal{B}_1$ which includes Boolean combinations of the following conditions $a_{1_1}, a_{1_2}, ..., a_{1_{11}}$:

a_{1_1} : cohabiting, a_{1_2} : housekeeping in common, a_{1_3} : having children in common, a_{1_4} : having sexual intercourse, a_{1_5} : having confirmed the relation by a contract, a_{1_6} : living in emotional fellowship, a_{1_7} : being faithful, a_{1_8} : giving mutual support, a_{1_9} : sharing economic assets and debts, $a_{1_{10}}$: having no legal impediments to marriage, $a_{1_{11}}$: having no similar relationship to another person.

Let us suppose that the consequences of having a relationship similar to being married are among the Boolean combinations of conditions $a_{3_1}, ..., a_{3_5}$ belonging to the domain B_3 of stratum $\mathcal{B}_3$.

We assume that in the *Bjs-triple* Ψ, $a_2 \in B_2$ is an intervenient between $(a_{1_1} \wedge a_{1_2} \wedge ... \wedge a_{1_{11}}) \in B_1$ and $(a_{3_1} \wedge ... \wedge a_{3_5}) \in B_3$, i.e.,

$$a_2 \curvearrowright \langle (a_{1_1} \wedge a_{1_2} \wedge ... \wedge a_{1_{11}}), (a_{3_1} \wedge ... \wedge a_{3_5}) \rangle.$$

Thus we assume that in the *Bjs-triple* Ψ, the conjunction $a_{1_1} \wedge a_{1_2} \wedge ... \wedge a_{1_{11}}$ is the weakest ground in B_1 for a_2 and $a_{3_1} \wedge ... \wedge a_{3_5}$ is the strongest consequence in B_3 of a_2.

Next we consider a *Bjs-triple* $\Psi^* = \langle \mathcal{S}_1^*, \mathcal{S}_2, \mathcal{S}_3^* \rangle$ where $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ are the same as in Ψ and where $\mathcal{S}_2$ remains unchanged but where $J_{1,2}^*$ and $J_{1,3}^*$ in Ψ^* are different from $J_{1,2}$ and $J_{1,3}$ in Ψ. We assume that $\mathcal{S}_1^* = \langle \mathcal{B}_1, \mathcal{B}_2, J_{1,2}^* \rangle$ and $\mathcal{S}_3^* = \langle \mathcal{B}_1, \mathcal{B}_3, J_{1,3}^* \rangle$ in Ψ^* are different from $\mathcal{S}_1$ and $\mathcal{S}_3$ in Ψ since in Ψ^*,

$$a_2 \curvearrowright \langle (a_{1_1} \wedge a_{1_2} \wedge a_{1_9} \wedge a_{1_{11}}), (a_{3_1} \wedge ... \wedge a_{3_5}) \rangle.$$

Thus in Ψ^*, the conjunction of $a_{1_1} \wedge a_{1_2} \wedge a_{1_9} \wedge a_{1_{11}}$ is the weakest ground for a_2. This means that in Ψ^*, the weakest ground for a_2 is the conjunction of:

a_{1_1} : cohabiting, a_{1_2} : housekeeping in common, a_{1_9} : sharing economic assets and debts, $a_{1_{11}}$: having no similar relationship to another person.

Obviously, in both Ψ and Ψ^* it holds that $(a_{1_1} \wedge a_{1_2} \wedge ... \wedge a_{1_{11}}) R_1 (a_{1_1} \wedge a_{1_2} \wedge a_{1_9} \wedge a_{1_{11}})$. Therefore, the joining $\langle (a_{1_1} \wedge a_{1_2} \wedge a_{1_9} \wedge a_{1_{11}}), a_2 \rangle$ in $J_{1,2}^*$ is narrower than the joining $\langle (a_{1_1} \wedge a_{1_2} \wedge ... \wedge a_{1_{11}}), a_2 \rangle$ in $J_{1,2}$. Accordingly, it also holds that the joining $\langle (a_{1_1} \wedge a_{1_2} \wedge a_{1_9} \wedge a_{1_{11}}), (a_{3_1} \wedge ... \wedge a_{3_5}) \rangle$ in $J_{1,3}^*$ is narrower than the joining $\langle (a_{1_1} \wedge a_{1_2} \wedge ... \wedge a_{1_{11}}), (a_{3_1} \wedge ... \wedge a_{3_5}) \rangle$ in $J_{1,3}$. We describe the situation by saying that the intervenient a_2 is ground-narrower in Ψ^* than in Ψ. This means that the weakest ground for a_2 in Ψ^* is less restricted than in Ψ.

In general terms we can say: If $\Psi = \langle \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \rangle$, $\Psi^* = \langle \mathcal{S}_1, \mathcal{S}_2^*, \mathcal{S}_3^* \rangle$ are *Bjs-triples* with $\mathcal{B}_i = \mathcal{B}_i^*$ $(1 \leq i \leq 3)$ and $a_2 \curvearrowright \langle a_1, a_3 \rangle$ in Ψ, $a_2 \curvearrowright \langle b_1, a_3 \rangle$ in Ψ^* and $\langle b_1, a_3 \rangle \lhd \langle a_1, a_3 \rangle$, then a_2 is ground-narrower in Ψ^* than in Ψ.[24]

[24] In [Lindahl and Odelstad, 2008a, sect. 3.5.1], we discuss the narrowing of "relationship similar to being married" with a different framework and terminology.

As an illustrative elaboration of the example, let us consider a normative system such as "Swedish law" as a class of *Bjs-triples* Ψ, Ψ^*, Ψ^{**}... etc. Then we might think of Ψ as a representation of "established Swedish law" and of Ψ^*, Ψ^{**}... etc. as developments of Ψ, made by new authoritative court decisions. Referring to the example, a new court decision resulting in Ψ^* still respects the established law in Ψ insofar as the joining $\langle(a_{1_1} \wedge a_{1_2} \wedge a_{1_9} \wedge a_{1_{11}}), a_2\rangle$ in established law Ψ still remains in system Ψ^*.

The possibility of narrowing an intervenient while respecting established law Ψ can be barred by a stipulation in Ψ that a certain combination of elements in B_1 is *not* a ground for the intervenient a_2. As regards the handling of this case, see [Odelstad and Lindahl, 2002, sect. 3.4] (cf. [Lindahl and Odelstad, 1999b]).

If we say that "relationship similar to being married" is an "open" concept in Swedish law, this might be taken to mean that established law in Ψ represents only a part of what is considered to count as Swedish law, and that Ψ^* is a development of the first regulative step taken by establishing Ψ.

Consequence-narrowing

What has been said about ground-narrowing has an analogous application in consequence-narrowing. The outlines of an example might regard the consequences of the intervenient *being the owner of an estate*. Let $\Psi = \langle\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3\rangle$ be a *Bjs-triple* with

$$\mathcal{S}_1 = \langle\mathcal{B}_1, \mathcal{B}_2, J_{1,2}\rangle,\ \mathcal{S}_2 = \langle\mathcal{B}_2, \mathcal{B}_3, J_{2,3}\rangle,\ \mathcal{S}_3 = \langle\mathcal{B}_1, \mathcal{B}_3, J_{1,3}\rangle,$$

with

a_2: x is the owner of an estate,

and where in Ψ it holds that a_2 is an intervenient between the disjunction $(a_{1_1} \vee a_{1_2} \vee ... \vee a_{1_m})$ of grounds for ownership and the conjunction $(a_{3_1} \wedge a_{3_2} \wedge ... \wedge a_{3_n})$ of consequences of ownership, i.e., where in Ψ it holds that

$$a_2 \curvearrowright \langle(a_{1_1} \vee a_{1_2} \vee ... \vee a_{1_m}), (a_{3_1} \wedge a_{3_2} \wedge ... \wedge a_{3_n})\rangle$$

Let $a_{3_{n+1}}$ be a consequence that is not a conjunct in the conjunction $(a_{3_1} \wedge a_{3_2} \wedge ... \wedge a_{3_n})$; for example let $a_{3_{n+1}}$ be the condition

$a_{3_{n+1}}$: x is permitted to erect a barbed-wire fence around the entire estate preventing others from entering.

In Ψ^* we have instead

$$a_2 \curvearrowright \langle(a_{1_1} \vee a_{1_2} \vee ... \vee a_{1_m}), (a_{3_1} \wedge a_{3_2} \wedge ... \wedge a_{3_n} \wedge a_{3_{n+1}})\rangle$$

where $a_{3_{n+1}}$ is a conjunct in the conjunction of consequences.

Since $(a_{3_1} \wedge a_{3_2} \wedge ... \wedge a_{3_n} \wedge a_{3_{n+1}})$ R_3 $(a_{3_1} \wedge a_{3_2} \wedge ... \wedge a_{3_n})$, it follows that the joining $\langle a_{2,}(a_{3_1} \wedge a_{3_2} \wedge ... \wedge a_{3_n} \wedge a_{3_{n+1}})$ which is narrowest in Ψ^* for the consequences of a_2 is narrower than the joining $\langle a_2, (a_{3_1} \wedge a_{3_2} \wedge ... \wedge a_{3_n})\rangle$ which is narrowest in Ψ. In this sense, the intervenient a_2 is consequence-narrower in Ψ^* than in Ψ. This means that the strongest consequence of a_2 in Ψ^* is richer than in Ψ.

In general terms: If $\Psi = \langle \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3\rangle$, $\Psi^* = \langle \mathcal{S}_1, \mathcal{S}_2^*, \mathcal{S}_3^*\rangle$ are *Bjs-triples* with $\mathcal{B}_i = \mathcal{B}_i^*$ $(1 \leq i \leq 3)$ and the joinings in Ψ, Ψ^* differ insofar as $a_2 \curvearrowright \langle a_1, a_3\rangle$ in Ψ, $a_2 \curvearrowright \langle a_1, b_3\rangle$ in Ψ^* where $\langle a_1, b_3\rangle \lhd \langle a_1, a_3\rangle$, then a_2 is consequence-narrower in Ψ^* than in Ψ.

What was said in the previous subsection of a normative system such as "Swedish law" as a class of *Bjs-triples* Ψ, Ψ^*, Ψ^{**}... and of developing established law by narrowing an intervenient applies to consequence-narrowing in an analogous way.

5.2.3 A legal illustration of a network of strata

The present subsection (with Figure 23 on page 621) presents a *cis* example of joining-systems with intervenients for a network of *Bqo* strata. (Cf. [Lindahl and Odelstad, 2011]) The example is legal and concerns *ownership* and *trust* as intervenients. The legal rules in this example are expressed in terms of joinings between *Bqo*'s $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_4$, $\mathcal{B}_5$ for ownership, and between $\mathcal{B}_3$, $\mathcal{B}_4$ and $\mathcal{B}_5$ for trusteeship.[25] Both of $\mathcal{B}_2$ and $\mathcal{B}_4$ are intermediate structures, where B_4 is supposed to contain the intervenients ownership and trusteeship and B_2 the intervenients *purchase*, *barter, inheritance*, *occupation*, *specification*, *expropriation* (for public purposes or for other reasons), which are grounds for ownership. B_1 contains grounds for the conditions in B_2, such as making a contract for purchase or barter respectively, having particular kinship relationship to a deceased person, appropriating something not owned, creating a valuable thing out of worthless material, getting a verdict on disappropriation of property, either for public purposes or for other reasons. B_3 contains different grounds for trusteeship. B_5 contains the legal consequences of ownership and trusteeship, respectively, in terms of powers, permissions and obligations.

The example is a useful illustration in several ways. Thus it illustrates a TJS representation of a fairly complex normative system. Also, as will be shown in the nest subsection, it illustrates various properties of intervenients in terms of minimality.

[25] Trust is where a person (trustee) is made the nominal owner of property to be held or used for the benefit of another. Trusteeship is the legal position of a trustee.

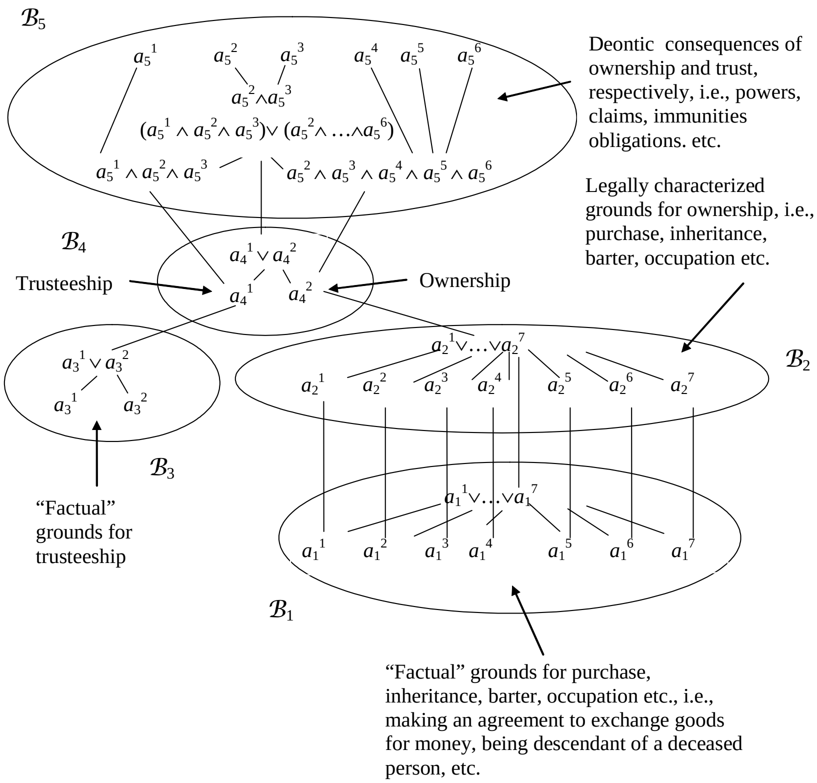


Figure 23

5.2.4 The typology of intervenient-minimality

The previous sections illustrate the role of intervenient concepts in the representation of a normative system. Of special interest is where intervenients exhibit different kinds of *minimality*. (To the following, see [Lindahl and Odelstad, 2011, pp. 132ff.]) Above, we have underlined the central role of minimal joinings and the formal structure of the set of minimal joinings. The previous sections provide tools for distinguishing between different kinds of intervenient minimality. We presuppose a *Bjs-triple* $\langle \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 \rangle$ in the sense of Definition 5.1 and non-degenerated intervenients in the sense of Definition 5.2.

If $a_2 \in \mathrm{Iv}\,(B_2, B_1, B_3)$ and $a_2 \curvearrowright \langle a_1, a_3 \rangle$, we say that,

a_2 is *correspondence-minimal* if $\langle a_1, a_3\rangle \in \min J_{1,3}$,
a_2 is *ground-minimal* if $\langle a_1, a_2\rangle \in \min J_{1,2}$,
a_2 is *consequence-minimal* if $\langle a_2, a_3\rangle \in \min J_{2,3}$.

Combining the three cases,

(1) $\langle a_1, a_3\rangle \in \min J_{1,3}$,

(2) $\langle a_1, a_2\rangle \in \min J_{1,2}$,

(3) $\langle a_2, a_3\rangle \in \min J_{2,3}$,

with their negations $(1'), (2'), (3')$, eight (2^3) cases are obtained. In the case $(1')\&(2')\&(3')$, the intervenient a_2 will be called *non-minimal*.

Not all eight cases are possible to realize. If $J_{1,3} = J_{1,2}|J_{2,3}$, then (1) is implied by (2)&(3). Hence, under this supposition, the case $(1')\&(2)\&(3)$ is impossible to realize.

As regards the importance of minimality emphasized above, note that the following holds: Suppose that $X_2 \subseteq B_2$ is such that for any $\langle x_1, x_3\rangle \in \min J_{1,3}$ there is $x_2 \in X_2$ such that $x_2 \curvearrowright \langle x_1, x_3\rangle$. Then

$$J_{1,3} = \{\langle a_1, a_3\rangle \in B_1 \times B_3 \mid \exists b_2 \in X_2 : \langle a_1, b_2\rangle \in J_{1,2} \text{ and } \langle b_2, a_3\rangle \in J_{2,3}\}.$$

Hence, a set of correspondence-minimal intervenients can be a convenient way for characterizing a set of joinings.

However, intervenients can be useful even if they are not correspondence-minimal. A type worth considering is $(1')\&(2)\&(3')$, i.e., where a_2 is ground-minimal but neither correspondence-minimal nor consequence-minimal. For instance, murder and high treason can have the same legal consequence (life imprisonment) notwithstanding that these crimes have different grounds.[26] Thus let

a_1 : grounds for murder, b_1: grounds for high treason
a_2 : murder, b_2 : high treason,
a_3 : life imprisonment

The example is illustrated by Figure 24.

We have $a_2 \curvearrowright \langle a_1, a_3\rangle$, $b_2 \curvearrowright \langle b_1, a_3\rangle$, $a_2 \vee b_2 \curvearrowright \langle a_1 \vee b_1, a_3\rangle$. The intervenient $a_2 \vee b_2$ is correspondence-minimal as well as ground- and consequence-minimal. Each of the intervenients a_2 and b_2, however, though ground-minimal, is neither consequence-minimal nor correspondence-minimal.

[26] See also [Lindahl and Odelstad, 2008a, sect. 3.2], for the case of “Boche” in the “Boche-Berserk” example. “Boche” and “Berserk” have different grounds but the same consequence.

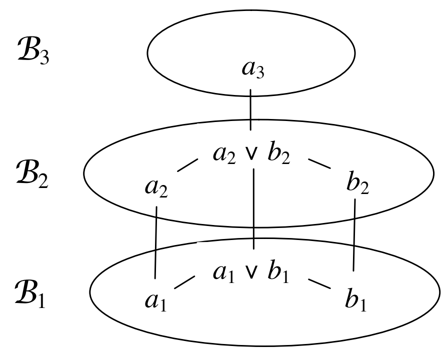


Figure 24

Types of intervenient minimality in the ownership/trust example

The ownership/trust example (Figure 23 on page 621) can be used for illustrating some types of intervenient minimality.

1. a_2^i $(1 \leq i \leq 7)$ is an intervenient from B_1 to B_4. This holds since WG $\left(a_1^i, a_2^i, B_1\right)$ and SC $\left(a_4^2, a_2^i, B_4\right)$ and hence $a_2^i \curvearrowright \left\langle a_1^i, a_4^2\right\rangle$. Note that (it is assumed that) $\left\langle a_1^i, a_2^i\right\rangle \in \min J_{1,2}$. Hence, the intervenient a_2^i is ground-minimal. However, a_2^i is neither correspondence-minimal (since $\left\langle a_1^i, a_4^2\right\rangle \notin \min J_{1,4}$), nor consequence-minimal (since $\left\langle a_2^i, a_4^2\right\rangle \notin \min J_{2,4}$).

2. $a_2^1 \vee ... \vee a_2^7$ is an intervenient from B_1 to B_4. This holds since

$$\text{WG}\left(a_1^1 \vee ... \vee a_1^7, a_2^1 \vee ... \vee a_2^7, B_1\right)$$

and

$$\text{SC}\left(a_4^2, a_2^1 \vee ... \vee a_2^7, B_4\right)$$

and hence

$$a_2^1 \vee ... \vee a_2^7 \curvearrowright \left\langle a_1^1 \vee ... \vee a_1^7, a_4^2\right\rangle.$$

It is assumed that

$$\left\langle a_1^1 \vee ... \vee a_1^7, a_2^1 \vee ... \vee a_2^7\right\rangle \in \min J_{1,2}$$

and that $\left\langle a_2^1 \vee ... \vee a_2^7, a_4^2\right\rangle \in \min J_{2,4}$. It then follows that $\left\langle a_1^1 \vee ... \vee a_1^7, a_4^2\right\rangle \in \min J_{1,4}$. (See the remark at the end of Section 3.7.) Hence, $a_2^7 \vee ... \vee a_2^7$ is ground-, consequence- and correspondence-minimal.

3. a_4^2 (being owner) is an intervenient from B_2 to B_5. This holds since

$$\mathrm{WG}\left(a_2^1 \vee ... \vee a_2^7, a_4^2, B_2\right)$$

and $\mathrm{SC}\left(a_5^2 \wedge ... \wedge a_5^6, a_4^2, B_5\right)$ and hence

$$a_4^2 \curvearrowright \left\langle a_2^1 \vee ... \vee a_2^7, a_5^2 \wedge ... \wedge a_5^6\right\rangle .$$

It is assumed that $\left\langle a_2^1 \vee ... \vee a_2^7, a_4^2\right\rangle \in \min J_{2,4}$ and

$$\left\langle a_4^2, a_5^2 \wedge ... \wedge a_5^6\right\rangle \in \min J_{4,5}.$$

It follows that

$$\left\langle a_2^1 \vee ... \vee a_2^7, a_5^2 \wedge ... \wedge a_5^6\right\rangle \in \min J_{2,5}.$$

Hence, the intervenient a_4^2 is ground-, consequence- and correspondence-minimal.

4. a_4^1 (being trustee) is an intervenient from B_3 to B_5. This holds since

$$\mathrm{WG}\left(a_3^1 \vee a_3^2, a_4^1, B_3\right)$$

and $\mathrm{SC}\left(a_5^1 \wedge a_5^2 \wedge a_5^3, a_4^1, B_5\right)$ and hence

$$a_4^1 \curvearrowright \left\langle a_3^1 \vee a_3^2, a_5^1 \wedge a_5^2 \wedge a_5^3\right\rangle .$$

It is assumed that $\left\langle a_3^1 \vee a_3^2, a_4^1\right\rangle \in \min J_{3,4}$ and that

$$\langle a_4^1, a_5^1 \wedge a_5^2 \wedge a_5^3\rangle \in \min J_{4,5}.$$

Once more it follows that

$$\left\langle a_3^1 \vee a_3^2, a_5^1 \wedge a_5^2 \wedge a_5^3\right\rangle \in \min J_{3,5}.$$

Hence, a_4^1 is ground-, consequence- and correspondence-minimal. On the other hand, since

$$\left\langle a_4^1 \vee a_4^2, a_5^2 \wedge a_5^3\right\rangle \in J_{4,5}$$

it follows that $\left\langle a_4^1, a_5^2 \wedge a_5^3\right\rangle \notin \min J_{4,5}$.

5. a_4^2 (being an owner) is an intervenient from $\mathcal{B}_1$ to $\mathcal{B}_5$. (Cf. 3 above.) This holds since

$$\mathrm{WG}\left(a_1^1 \vee ... \vee a_1^7, a_4^2, B_1\right)$$

and

$$\mathrm{SC}\left(a_5^2 \wedge ... \wedge a_5^6, a_4^2, B_5\right)$$

and hence

$$a_4^2 \curvearrowright \left\langle a_1^1 \vee ... \vee a_1^7, a_5^2 \wedge ... \wedge a_5^6 \right\rangle .$$

Here, it is assumed that (i)

$$\langle a_1^1 \vee ... \vee a_1^7, a_2^1 \vee ... \vee a_2^7 \rangle \in \min J_{1,2},$$

that (ii)

$$\langle a_2^1 \vee ... \vee a_2^7, a_4^2 \rangle \in \min J_{2,4}$$

and that (iii)

$$\langle a_4^2, a_5^2 \wedge ... \wedge a_5^6 \rangle \in \min J_{4,5}.$$

From (iii) it follows that a_4^2 is consequence minimal. From (i)-(iii) and (once more) the remark in Section 3.7 it follows that $\langle a_1^1 \vee ... \vee a_1^7, a_4^2 \rangle \in \min J_{1,4}$ (ground minimality for a_4^2), and that

$$\langle a_1^1 \vee ... \vee a_1^7, a_5^2 \wedge ... \wedge a_5^6 \rangle \in \min J_{1,5}$$

(correspondence minimality for a_4^2). Hence, a_4^2 is ground-, consequence- and correspondence minimal.

6 Related work

6.1 Previous work of ours

In our first major joint work on the subject of intermediate concepts, viz. [Lindahl and Odelstad, 1999a], we presented a number of ideas to be further developed in subsequent papers of ours.[27] Our concern with intermediaries originally was inspired by the Scandinavian legal and philosophical discussion of intermediate concepts in the law, a discussion started in the 1940's by Ekelöf and Wedberg. Other sources of inspiration were Dummett's theory of language, Gentzen's theory of natural deduction and the theory of normative systems of Alchourrón and Bulygin. (See Section 1.7 above.)

Our aim in [Lindahl and Odelstad, 1999a] was to provide tools for a rational reconstruction of a legal system with intermediaries; the formal framework was that of a *lattice* $\langle L, \leq \rangle$ of conditions and an implicative relation $\wp$ over L such that $\langle L_\wp, \leq_\wp \rangle$ is generated by the equivalence relation

[27] [Lindahl and Odelstad, 1999a] was based on our presentation at DEON'98 in Bologna. Our joint theory was presented for the first time in 1996 at the workshop (a cura di V. A. A. Martino) *Logica, Informatica, Diritto*, Pisa, 1996, in honor of Carlos Alchourrón. For references to another preparatory joint work in 1996 see [Lindahl and Odelstad, 1999a]. An early paper in Swedish by Lindahl on intermediate concepts is [Lindahl, 1985].

$\approx_\wp$. Within this framework, we defined the notion of a lattice joining-system $\langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$, with $\mathbf{A}$ the under-lattice, $\mathbf{B}$ the over-lattice and $\mathbf{C}$ the background lattice. We defined two kinds of linking relations between $\mathbf{A}$ and $\mathbf{C}$, viz. the relations of "connection" and "coupling". We treated themes such as couplings satisfying a constraint, the relations "narrower" and "wider" for couplings, and the interrelation between coupling conditions and the notion of "intermediary".

In subsequent papers, we exchanged the main framework of lattices for a framework of *Boolean quasi-orderings* (*Bqo*'s, cf. Section 4.1.1 above.)[28] Connections and couplings now were thought of as relations between what we called "fragments" of a *Bqo*. A *Bqo* $\langle B, \wedge, ', R \rangle$ was thought of as the "closure" of a supplemented Boolean algebra $\langle B, \wedge, ', \rho \rangle$.[29] Also, the algebraic framework was made more abstract, so as to consider "condition implication structures" as models of the more abstract framework. Within this framework, the theory was further developed in various respects. In [Lindahl and Odelstad, 1999b], we introduced the idea of a normative system as a set of *Bqo*'s, among which a "core" and a number of "amplifications"; in [Lindahl and Odelstad, 2000], we treated the problem of intermediate legal concepts that (like disposition concepts) express hypothetical consequences; in [Odelstad and Lindahl, 2002], we further developed the theory of connections; in [Lindahl and Odelstad, 2003], we treated the idea of subtraction and addition of norms; in [Lindahl and Odelstad, 2004], we proposed a model for normative positions within the algebraic framework; and, in [Lindahl and Odelstad, 2006b], we dealt summarily with open and closed intermediaries.

A third stage of development with regard to the general framework appeared with the introduction of *Boolean joining-systems* (*Bjs*'s, cf. above Section 4), first presented in [Odelstad and Boman, 2004]. Instead of considering connections and couplings between two fragments of one single *Bqo*, we now introduced the idea of a *Bjs* $\langle \mathcal{B}_1, \mathcal{B}_2, J \rangle$ with a joining relation J from one *Bqo* $\mathcal{B}_1$ to another *Bqo* $\mathcal{B}_2$. We adjusted the analyses of the issues mentioned above to this framework and developed new themes. In particular, in [Lindahl and Odelstad, 2006a], we introduced the notion of "intervenient" as a formal tool for analyzing intermediaries in normative systems and began the development of a formal theory of intervenients. The theory of intervenients was further developed in [Lindahl and Odelstad, 2008a] and included topics such as "bases of intervenients", "extendable and non-extendable intervenients", and negations of intervenients. The formal

[28] The idea of Boolean quasi-orderings and fragments was first presented already in 1998, see references in [Odelstad and Lindahl, 2000].

[29] Cf. [Lindahl and Odelstad, 1999b].

analysis of intervenients was continued in [Lindahl and Odelstad, 2008b; Lindahl and Odelstad, 2011]. The focus of the latter paper is on intervenient minimality, conjunctions and disjunctions of intervenients, organic wholes of intervenients, and a typology of different kinds of intervenients. Also [Lindahl and Odelstad, 2011] pays attention to the properties of intervenients in a network of several *Bjs*'s, with "strata" of *Bqo*'s $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \ldots$.

6.2 Recent work of others

6.2.1 A remark on the "Counts-as" theory

A logical analysis of external sentences of the kind "x counts-as y in s", where s is an institution (s can be a normative system), was proposed by Jones and Sergot in [Jones and Sergot, 1996; Jones and Sergot, 1997]. The work of Jones and Sergot on "Counts-as" has been continued by a number of other authors. This subsequent work has many facets, developed over the past ten years. The book-length study [Grossi, 2007] by Grossi provides axiomatization and semantics of the different counts-as operators.

When a rule r of a legal system $\mathcal{N}$ attaches an intermediary m, e.g., "x and y have made a contract to the effect that z", to a conjunction a of facts, the rule r can be expressed in different ways, e.g. "if a then m", "a is a ground for m" or, sometimes, "a counts as m".

As appears from the foregoing, in our formal representation of $\mathcal{N}$ by a *cis* model of *Bjs-triples* $\langle \mathcal{S}_i, \mathcal{S}_j, \mathcal{S}_k \rangle$ we represent such a statement by $a_i R_i b_i$, or (if different sorts of objects are in view) $a_i J_{i,j} a_j$, which statements are read "a_i implies b_i" and "a_i is a ground for a_j" respectively. In TJS, no counts-as operator is introduced, and in the present chapter we do not examine the question in which cases the counts-as vocabulary might be appropriate. Rather, referring to the joint paper [Grossi *et al.*, 2007] by Grossi, Meyer and Dignum, we will be content, by an example, merely to suggest how some of the material dealt with in the Counts-as theory might be represented in our theory. (Cf. [Lindahl and Odelstad, 2008a, sect. 3.5.3].)

In [Grossi *et al.*, 2007, p. 2], the following example is given of three kinds of Counts-as:

> "It is a rule of normative system Γ that conveyances transporting people or goods count as vehicles; it is always the case that bikes count as conveyances transporting people or goods but not that bikes count as vehicles; therefore, in the context of normative system Γ, bikes count as vehicles."

According to [Grossi *et al.*, 2007, p. 2], the first premise states a rule of Γ and is a constitutive Counts-as, the second premise states a generally acknowledged classification, thus states a general classificatory Counts-as, and the conclusion states a classification that holds in Γ and is a Counts-as brought about by Γ though it is not a constitutive Counts-as.

The example can be further developed by the assumption that in Γ vehicles are not admitted in public parks (cf. [Grossi *et al.*, 2006, p. 615]).

If counts-as sentences are seen as internal to a normative system Γ, a representation of the example might be made in terms of Figure 25 on page 628. We can conceive of the example in such a way that "being a

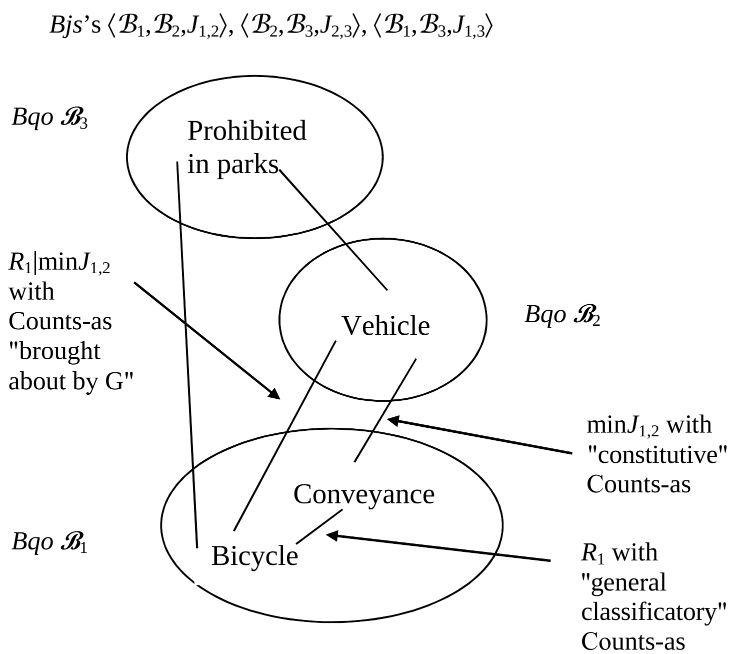


Figure 25

vehicle" is an intervenient from B_1 to B_3 corresponding to the pair $\langle$being a conveyance, being prohibited in parks$\rangle$ in $B_1 \times B_3$.

In this chapter, there is no room for going into possible developments of the example. A brief comment should be made, however, on how we might represent something similar to the distinction between three kinds of Counts-as made by Grossi, Meyer and Dignum. We can assume that relation R_1 (a subset of $B_1 \times B_1$) represents implications that hold in an uncontroversial way independently of the instituted rules of Γ. In contrast, the set of minimal joinings $\min J_{1,2}$ (a subset of $B_1 \times B_2$) can be seen as expressing implications that are instituted by the rules in Γ. If this view is taken, distinctions can be made as follows. (We write b, c, v for "bicycle", "conveyance", "vehicle".) Firstly, the general classification of bicycles as conveyances is due to $\langle b, c\rangle \in R_1$ ("bikes always count as conveyances"). Secondly, the classification of conveyances as vehicles is due to $\langle c, v\rangle \in \min J_{1,2}$ ("Conveyances ... are to count as vehicles"). Thirdly, the classification of bicycles as vehicles is due to $\langle b, v\rangle \in R_1 | \min J_{1,2}$ (the relative product).

6.2.2 Input-output logic

In a series of papers, Makinson and van der Torre have developed a theory called input-output logic, see for example [Makinson and van der Torre, 2000; Makinson and van der Torre, 2003]. Important similarities between input-output logic and our approach are that we study normative systems as deductive mechanisms yielding outputs for inputs and that norms are represented as ordered pairs.[30] Other similarities worth mentioning are that neither the principal output operation in input-output logic, nor the relation J in a *Bjs*, requires reflexivity or contraposition.

TJS, however, differs from input-output logic, as developed in [Makinson and van der Torre, 2000; Makinson and van der Torre, 2003], in a number of respects. Thus, in TJS,

1. if a pair $\langle a_1, a_2\rangle$ represents a norm, this is due to the normative character of a_2 (see Sections 1 and 4.4);

2. a central theme is "intermediaries" (intermediate concepts) in the system;

3. a normative system is represented as a network of subsystems and relations between them; the study comprises stratification of a normative system with structures ("strata") that are intermediate;

4. emphasis is put on the analysis of minimality of joinings and of closeness between strata; representation by a base of minimal joinings is of special importance;

5. the strata of the kind of system called a Boolean joining-system are Boolean structures (*Bqo's* to be more precise); however, the strata of joining-systems of other kinds need not in TJS be Boolean structures. Thus, in Section 3 of the present chapter, there is a general algebraic

[30] Cf. [Lindahl and Odelstad, 1999b, sect. 1.1], with a reference to the work of Alchourrón and Bulygin in [Alchourrón and Bulygin, 1971].

framework for joining-systems that need not be Boolean, for example joining-systems containing strata of lattice-like structures. (In input-output logic, the set of inputs constitutes a Boolean algebra and the same holds for the set of outputs.)

The following remark sheds some light on the relation between input-output logic and the theory of joining-systems. Suppose that $\langle \mathcal{B}_1, \mathcal{B}_2, J\rangle$ is a *Bjs* where $\mathcal{B}_1 = \langle B_1, \wedge, ', R_1\rangle$ and $\mathcal{B}_2 = \langle B_2, \wedge, ', R_2\rangle$. Makinson and van der Torre state a number of rules for the so-called "basic" output operator (called out_2) that they define. Translated to a *Bjs* these rules are as follows (cf. Definitions 3.11 in Section 3.2):

Strengthening Input: From $\langle a_1, a_2\rangle \in J$ to $\langle b_1, a_2\rangle \in J$ whenever $b_1 R_1 a_1$.
Follows from condition (1) of a *Bjs*.
Conjoining Output: From $\langle a_1, a_2\rangle \in J$ and $\langle a_1, b_2\rangle \in J$ to $\langle a_1, a_2 \wedge b_2\rangle \in J$.
Follows from condition (3) of a *Bjs*.
Weakening Output: From $\langle a_1, a_2\rangle \in J$ to $\langle a_1, b_2\rangle \in J$ whenever $a_2 R_2 b_2$.
Follows from condition (1) of a *Bjs*.
Disjoining Input: From $\langle a_1, a_2\rangle \in J$ and $\langle b_1, a_2\rangle \in J$ to $\langle a_1 \vee b_1, a_2\rangle \in J$.
Follows from condition (2) of a *Bjs*.

There are three conditions on a joining space in a Boolean joining-system. The comparison with input-output logic above shows that it could be of interest to define weaker kinds of systems characterized by, for example, conditions (1) and (3).

In TJS the notion of completeness plays an important role. If in a joining-system the quasi-orderings are complete quasi-lattices, then the joining-system satisfies connectivity, one of the key feature in TJS. Even in the definition of a joining-system itself, the notion of completeness is in some sense present although in a concealed form. To see this, we recall condition (2) and (3) in the definition of a joining-system. In these conditions, least upper bounds (lub's) and greatest lower bounds (glb's) of arbitrary sets are called for, although such bounds are not required to exist. Instead certain things must hold for those lub's or glb's of infinite sets that exist. Admittedly, however, this may in certain contexts be regarded as too demanding a requirement: if so, it may seem reasonable to restrict attention to lub's and glb's of pairs of objects. This reasoning leads to the following definition of a kind of systems called prejoining-systems.

Definition 6.1 *A* prejoining-system*, is an ordered triple $\langle \mathcal{A}_1, \mathcal{A}_2, J\rangle$ such that $\mathcal{A}_1 = \langle A_1, R_1\rangle$ and $\mathcal{A}_2 = \langle A_2, R_2\rangle$ are quasi-orderings and $J \subseteq A_1 \times A_2$ and the following conditions are satisfied where $\trianglelefteq$ is the narrowness relation*

determined by $\mathcal{A}_1$ *and* $\mathcal{A}_2$*:*
(1) *for all* $b_1, c_1 \in A_1$ *and* $b_2, c_2 \in A_2$*, if* $\langle b_1, b_2\rangle \in J$ *and* $\langle b_1, b_2\rangle \trianglelefteq \langle c_1, c_2\rangle$*, then* $\langle c_1, c_2\rangle \in J$*,*
(2) *for all* $b_1, c_1 \in A_1$ *and* $b_2 \in A_2$*, if* $\langle b_1, b_2\rangle \in J$ *and* $\langle c_1, b_2\rangle \in J$*, then* $\langle a_1, b_2\rangle \in J$ *for all* $a_1 \in \mathrm{lub}_{R_1}\{b_1, c_1\}$*,*
(3) *for all* $b_2, c_2 \in A_2$ *and* $b_1 \in A_1$*, if* $\langle b_1, b_2\rangle \in J$ *and* $\langle b_1, c_2\rangle \in J$*, then* $\langle b_1, a_2\rangle \in J$ *for all* $a_2 \in \mathrm{glb}_{R_2}\{b_2, c_2\}$*.*

Connectivity is not so firmly connected with prejoining-systems as with TJS joining-systems. The reason is roughly that the occurrence of lub's and glb's of infinite sets fits well with quasi-orderings satisfying completeness in the sense of being complete quasi-lattices. The importance of connectivity in TJS has been stressed several times.

A brief note on the role of the notion of closure system in TJS is in order. An important aspect of TJS is that it gives a method for representing a set of conditional norms in an elaborated way. Suppose that $\mathcal{B}_1$ is a *Bqo* of grounds and $\mathcal{B}_2$ is a *Bqo* of consequences. Let us suppose that K is a set of conditional norms with the antecedents taken from B_1 and the consequences taken from B_2. Hence, $K \subseteq B_1 \times B_2$ and K is a correspondence from B_1 to B_2. K can be thought of as a "crude" representation of a normative system $\mathcal{N}$. Then, a set K^* can be generated by forming the "joining closure" of K such that $\langle \mathcal{B}_1, \mathcal{B}_2, K^*\rangle$ is a *Bjs*. This is an "elaborated" representation of $\mathcal{N}$.

The *out*-operations introduced by Makinson and van der Torre also use a closure-operation, viz. classical consequence. With some simplification one can say that Makinson and van der Torre form the closure of the input and of the output but leave the set of norms as it is. However, it turns out that, regarded only as deductive mechanisms, input-output logic and the theory of joining-systems give rather similar results in spite of their use of different closure-operations in different ways. As a conjecture we suggest the following. Suppose that the *Bqo*'s $\mathcal{B}_1$and $\mathcal{B}_2$ are Boolean algebras, i.e. for $i = 1, 2$, R_i is the partial ordering determined by the Boolean algebra $\langle B_i, \wedge, '\rangle$. Then $J = out_1(J)$. Furthermore, if $\mathcal{B}_1$ is a complete Boolean algebra and some general conditions are satisfied, then $J = out_2(J)$.

Acknowledgments

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[31] The chapter is the result of wholly joint work where the order of appearance of our author names has no significance.

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