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An Analysis of Kautz-Volterra models for modeling block structure nonlinear systems

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Abstract

Kautz-Volterra (KV) models of some nonlinear systems were analyzed. The relations between the true pole of the analyzed systems and the optimal pole of the KV models were analyzed. The properties of nonlinear systems depend on the order of the subsystems, since nonlinear operators do not commute. Wiener (H-N) and Hammerstein (N-H) systems were analyzed.

I Introduction

The Volterra theory has been widely used for analyzing nonlinear systems with memory [1, 2], and it is suitable for weakly nonlinear systems. A problem in system identification is that the number of parameters of a full Volterra model increases rapidly with the nonlinear order and memory depth. Behavioral models are thus, in many cases, reductions of a general Volterra model [1] and are often block structures of linear time invariant filters and memoryless nonlinearities. Models with a linear filter, H, followed by a nonlinearity, N, denoted a Wiener model (H-N), or vice versa, a Hammerstein model (N-H), are often proposed as standard nonlinear dynamic models [3]. H-N and N-H models and variations of these are often denoted "box-models" and are sub classes of the full Volterra model. Box-models are often easy to identify, but do not have the general properties of the Volterra model [1, 4].

A number of papers on nonlinear behavioral models based on orthonormal basis functions have been presented in recent years [5-7]. In these, the basis function expansion is made around one or several fixed poles. Models using orthonormal basis functions were first developed for linear systems where a small number of parameters was of importance, e.g., in control. The orthonormal basis functions are expansions around the dominant pole of the system, and a priori knowledge can therefore be used. For a real-valued pole, the orthonormal basis functions are Laguerre functions [5, 8]; for a complex pole the orthonormal basis functions are Kautz functions [6, 9]. Orthonormal basis functions that are expansions around more than one pole have also been derived [10].

In several papers, the identification of orthonormal basis functions in models of nonlinear dynamic systems has been investigated [5-7, 11, 12]. Nonlinear models using Kautz functions and with the same general properties of a Volterra model are denoted Kautz-Volterra (KV) models. In [13] block models of H-N and N-H systems with linear filters identified by orthonormal basis functions, the order of the linear filter and the static nonlinearity were assumed to be known.

There is a fundamental difference between linear and nonlinear systems that is important for KV models. Nonlinear operators do not commute, in contrast to linear ones. Thus, the total transfer function of an N-H system will be different from that of an H-N system, if N is a

nonlinear operator. The order of the subsystems has a large impact on the performance of the KV models, as will be shown in this paper.
In this paper we will investigate how the optimal pole of a KV model of some nonlinear systems modeled by simple block structures is related to the true pole. In particular, the N-H and H-N systems will be analyzed.

II Theory

We present all of the theory that is necessary to fully understand the essence of the paper. The theory is mainly presented in continuous time except for some parts, in which it is convenient to use discrete-time notation, e.g., the KV model (the original contribution with discrete-time notation) or when the mathematics can be more easily understood as in III Analysis.

A. Volterra theory

A nonlinear dynamic system with fading memory, H, can be approximated with a Volterra series [1, 2]

$$y(t) = \mathrm{H}\left[u(t)\right] \approx \int h_1(\tau)u(t-\tau)d\tau + \iint h_2(\tau_1,\tau_2)u(t-\tau_1)u(t-\tau_2)d\tau_1 d\tau_2 + ... \quad (1)$$

where $u(t)$ is the input signal, $y(t)$ is the output signal, and h_1, h_2 are the 1st, 2nd,.. order Volterra kernels.

The Volterra kernels are symmetric with respect to permutations of $\tau_1, \tau_2, ..., \tau_n$ [2], i.e.

$$h_n(\tau_1,\tau_2,...,\tau_n) = h_n(\tau_2,\tau_1,....,\tau_n). \quad (2)$$

In a time discrete form, the Volterra series becomes

$$y(n) \approx \sum h(m)u(n-m) + \sum\sum\iint h_2(m_1,m_2)u(n-m_1)u(n-m_2). \quad (3)$$

The number of parameters $h_i(m_1,m_2)$ increases as with factorial behavior [14], which makes it practically difficult to identify Volterra models.

B. The Kautz-Volterra model

Due to the drawback of the rapid increase of parameters in the Volterra series [1, 2], another nonlinear model with the same general properties as the Volterra series has recently aroused some interest, namely the KV model [14]. The model is evaluated for radio frequency power amplifier in [14]. The output $y_{KV}(t)$ of the KV model is

$$y_{KV}(t) = y_{KV}^{(1)} + y_{KV}^{(2)} + y_{KV}^{(3)} + \dots \quad (4)$$

where

$$y_{KV}^{(i)}(t) = \sum_{l_1=1}^{N_i}\sum_{l_2=l_1}^{N_i} \cdots \sum_{l_i=l_{i-1}}^{N_i} v_{l_1\cdots l_i} \times x_{i,l_1}(t)x_{i,l_2}(t)\dots x_{i,l_i}(t) \quad (5)$$

and $x_{i,l_k}(t)$ is the output of the kth filter in an orthonormal filter bank of N_i linear filters corresponding to order i. $v_{l_1 \ldots l_i}$ are coefficients for the respective filtered signals. The filters are excited with the input signal $u(t)$. See Fig. 1.

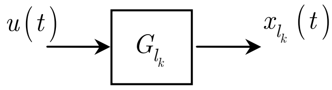

Fig. 1. The output of the kth filter excited with the input signal u.

The Kautz filters in the filter bank G can support the use of one (one complex pole-pair) or several poles (pole-pairs) per nonlinear order. If the originally Kautz functions as in [9, 15] are considered the Kautz functions are defined by

$$\begin{aligned}
\Psi_{2k-1}(s,b,c) &= \frac{\sqrt{2b}s}{s^2+bs+c}\left[\frac{s^2-bs+c}{s^2+bs+c}\right]^{k-1}, \\
\Psi_{2k}(s,b,c) &= \frac{\sqrt{2bc}}{s^2+bs+c}\left[\frac{s^2-bs+c}{s^2+bs+c}\right]^{k-1}, \\
& b>0, c>0, \quad k=1,2,\ldots,N
\end{aligned} \qquad (6)$$

and are often denoted the "two-parameter" Kautz functions. If the complex pole-pairs defined by (6) are close to the poles of the true system, the convergence is fast, using few parameters for a linear system.

A discrete formulation in which several poles can be included was presented in [10]. In [10], the filters are

$$\begin{aligned}
G_{i,l}(z) &= \sqrt{1-|\xi|_l^2}\,\frac{z}{(z-\xi_l)}\prod_{q=1}^{l-1}\frac{(1-\xi_q^* z)}{(z-\xi_q)}, \\
l &= 1,2,..,N
\end{aligned} \qquad (7)$$

where ξ_q^* denotes the complex conjugate of the pole ξ_q. It was shown in [10] that the construction in (7) preserves orthonormality. In Section III, the concept with more than one pole per nonlinear order in agreement with (7) is used to introduce *virtual poles*.

C. Model systems

Two important block system structures are investigated in this paper, i.e., the H-N and the N-H systems. The structures are shown in Fig. 2.

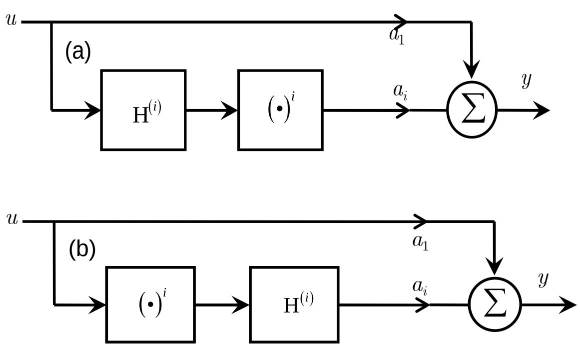

Fig. 2. The two model systems: (a) the H-N (Wiener) and (b) the N-H (Hammerstein) systems. The nonlinear order (i) is chosen to be 2 or 3 for the second- and third-order systems.

For an H-N system, the 2nd order kernel is [1]

$$h^{(2)}(\tau_1,\tau_2) = h(\tau_1)h(\tau_2) \qquad (8)$$

A 2nd order N-H system has the properties [1]

$$h^{(2)}(\tau_1,\tau_2) = h(\tau_1)\delta(\tau_1 - \tau_2) \qquad (9)$$

There are corresponding expressions for the higher-order kernels.

The output, $y_{2,W}$, of a 2nd order H-N system, as shown in Fig. 2(a), is

$$y_{2,W}\left(t\right) = a_2\left[\int_0^t h^{(2)}\left(\tau\right)u\left(t-\tau\right)d\tau\right]^2 \qquad (10)$$

where $h^{(2)}\left(t\right)$ is the impulse response of the linear filter $\mathrm{H}^{(2)}$. The total system output is then the sum of all the nonlinear branches included in the system, as illustrated in Fig. 2(a).

Using the "two-parameter" Kautz functions (6) in the KV model specified by (4), (5) and, as an example, considering only the first ($N = 1$) Kautz function per order, the third-order KV model output collapses into

$$\begin{aligned} y_{KV}\left(t\right) &= v_1\left[\int_0^t \psi_1\left(\tau\right)u\left(t-\tau\right)d\tau\right] \\ &+v_2\left[\int_0^t \psi_2\left(\tau\right)u\left(t-\tau\right)d\tau\right]^2 \\ &+v_3\left[\int_0^t \psi_3\left(\tau\right)u\left(t-\tau\right)d\tau\right]^3 \end{aligned} \qquad (11)$$

where

$$\begin{aligned} &\psi_1\left(t\right) = \sqrt{2}e^{-b_1\frac{t}{2}} \\ &\times\left[\sqrt{b_1}\cosh\left(\frac{t}{2}\sqrt{b_1^2-4c_1}\right) - \frac{b_1^{3/2}\sinh\left(\frac{t}{2}\sqrt{b_1^2-4c_1}\right)}{\sqrt{b_1^2-4c_1}}\right], \end{aligned} \qquad (12)$$

$$\psi_2\left(t\right) = \frac{\sqrt{8b_2c_2}e^{-b_2\frac{t}{2}}\sinh\left(\frac{t}{2}\sqrt{b_2{}^2-4c_2}\right)}{\sqrt{b_2{}^2-4c_2}}, \qquad (13)$$

and $\psi_3\left(t\right)$ similar to $\psi_1\left(t\right)$ so that the Laplace transforms of $\psi_1\left(t\right)$ and $\psi_2\left(t\right)$ are

$$\Psi_1\left(s\right) = \frac{\sqrt{2b_1}s}{s^2+b_1s+c_1} \qquad (14)$$

and

$$\Psi_2\left(s\right) = \frac{\sqrt{2b_2c_2}}{s^2+b_2s+c_2}, \qquad (15)$$

respectively, and $u\left(t\right)$ denotes the input signal.

The model parameters that minimize the square error, i.e.

$$\varepsilon = \int_0^t \left| y_W(\tau) - y_{KV}(\tau) \right|^2 d\tau, \qquad (16)$$

are identified and, specifically, the poles according to e.g., (14) and (15) are denoted as optimal. By inspection it is easily seen, by a comparison of (10) and (11), that (16) has its minimum when the poles in. $\mathrm{H}^{(2)}$ and Ψ_2 are identically positioned.

To approximate an N-H model system with the KV model is not trivial. In Section III, we present a mathematical treatment of the problem of finding a Volterra approximation as in (5) of the second-order N-H system described by the time-zero response $f(u) = H[u](0)$. We use the discrete time domain since the N-H system is quite delicate to define the distribution-valued white noise input in continuous time.

The second-order N-H system can be taken to represent a general quadratic form $u \to f(u)$ in the input u. In a way, the H-N system and the N-H system represent two extreme forms of the quadratic form, where the H-N system is one-dimensional from the onset, and the N-H system can be taken to represent any quadratic form of any dimension. The investigation shows that there is a fundamental limitation so that we cannot in general hope to use a small number of orthonormal basis functions for approximating the N-H system.

How well the system behaves under a Volterra approximation will depend on how concentrated the spectrum is of the compact symmetric operator corresponding to the quadratic form f. These spectral properties depend both on f and the input u. The results in Section III, Table I, where a one-dimensional second-order approximation is applied to a fixed N-H system f on band limited white noise, show band limitation effects of such a concentration spectrum.

III Analysis

TABLE II. EXPLANATION OF NOTATION IN THE APPENDIX

$u(t)$	A fixed Gaussian input process.
$y(t) = F[u](t)$	The response to input $u(t)$ by a general stationary causal (nonlinear) filter F. E.g., the output from a general second-order N-H system $H[u](t)$.
$f(u)$	The time-zero response $y(0)$. Any real stochastic variable determined by $u(-\tau), \tau > 0.$
$\left(L^2, \lVert\cdot\rVert_2\right)$	The Hilbert space of the family $\{f_\alpha\}$ of time-zero responses such that $\lVert f_\alpha \rVert_2^2 := E\left(\lvert f_\alpha \rvert^2\right) < \infty$. By stationarity and causality, this is equivalent to a corresponding family $\{F_\alpha\}$ of filters for a fixed input.
$H \subset L^2$	The Gaussian subspace of L^2 consisting of the time-zero response to linear filters. The closed linear span of the input $\{u(-\tau) : \tau > 0\}$.
$\{u_t\}_{t=1}^{\infty}$	The input process $u_t = u(-t), t \in \mathbb{Z}_{\geq 0}$, in discrete time. Possibly obtained after a change of basis in $H,$ which is supposed to have a countable basis.
$V = \langle \xi_j \rangle_{j=1}^{M}$	A finite dimensional subspace of H of dimension M. Corresponds to the linear span of responses $\{\xi_j\}_{j=1}^{M}$ to an orthonormal linear filter bank. E.g., a set of M orthonormal Kautz-filters.
$L^2(V)$	The subspace of $L^2 = L^2(H)$ of functions on V.

$\mathcal{P}_n(V)$	For a subspace $V \subset H$, the set of $n\text{th}$ degree polynomials with arguments in V. The Volterra approximation asks for a $p \in \mathcal{P}_n(V)$.
$\tilde{f}$	The best approximation of f in $L^2(V)$. (The orthogonal projection.) If $f \in \mathcal{P}_n(H)$, i.e., if f has a nonlinearity of order at most n, then $\tilde{f} \in \mathcal{P}_n(V)$, by the Wiener Chaos decomposition. E.g., $\tilde{f}$ is the time-zero response of the best Kautz-approximation $y_{KV}(t) = y_{KV}^{(1)}(t) + y_{KV}^{(2)}(t) + \ldots$
g_t	The diagonal of the kernel h_2 in the properly normalized N-H system: i.e., $h_2(t,s) = g_t \cdot \delta_{ts}$ and u_t is discrete white noise.

A1. A Discrete Formulation and a General Form of the Approximation Problem

Let $u_t, t \geq 0$, denote the "past" $u(t)$s at time zero, i.e., for $(u_t)_{t=1}^{\infty} = [u(-t)]_{t=1}^{\infty}$. Let $L^2 := L^2(\Omega, \mathcal{G}, \mathbb{P})$, where $\mathcal{G}$ is the sigma-algebra generated by $u_t s$. That is, L^2 is the Hilbert space of square integrable random variables that can be expressed as functions of the past Gaussian input (u_t). Any mean-square integrable stationary causal filter of the form $y = H(u)$ is, by stationarity, uniquely identified with the functional dependence at time zero, i.e., by the function

$$y(0) = f(u_1, u_2, \ldots) = f[u(-1), u(-2), \ldots], \quad (17)$$

where $f(u) \in L^2$. From such an f, we obtain the full process by translation, i.e., by taking $y(t) := f[u(t-1), u(t-2), \ldots]$. The Hilbert space L^2 has a *Gaussian subspace* H consisting of the functions $\xi = \xi(u)$, which are linear in u, i.e., functions that can be expressed as $\xi(u) = \sum_{t=1}^{\infty} \xi_t u_t$ for some square summable sequence $(\xi_t), \sum_t \xi_t^2 < \infty$. The space H is a *Gaussian Hilbert space* (see [16]) of jointly Gaussian variables. Since u is assumed to be discrete white noise input with $E(u_t u_s) = \delta_{ts}$, we obtain an isometry between H and the space of square summable sequences: **If** $\xi = \sum_t \xi_t u_t$ and $\tilde{\xi} = \sum_t \tilde{\xi}_t u_t$, then

$$\langle \xi, \tilde{\xi} \rangle_{L^2} = E(\xi \tilde{\xi}) = \sum_{t=1}^{\infty} \xi_t \tilde{\xi}_t. \quad (18)$$

For a closed linear subspace V of H, we define the linear subspace $L^2(V)$ of $L^2 = L^2(H)$ of square integrable functions of elements in V. Furthermore, let $\mathcal{P}_n(V) \subset L^2(V)$ denote the set of multivariable polynomials $p(\xi_1, \xi_2, \ldots, \xi_m)$ of degree less than or equal to n, where the variables ξ_k are elements of V. Note that this is a linear subspace of L^2, which is closed if and only if V is finite dimensional.

For the approximation, we consider a given physical system $f(u)$. The general Volterra approximation problem is to find a good approximation $\tilde{f}$ of f among the polynomials in

$\mathcal{P}_n(V)$, for some n and some finite dimensional subspace V, such that $\tilde{f}$ corresponds to a finite Volterra expansion of the input signal. By a *good approximation in* V *of order* n, we will mean an approximation $\tilde{f} \in \overline{\mathcal{P}}_n(V)$ where the mean square error $\|f - \tilde{f}\|_2^2$ is minimal. For a fixed V, we can usually find the optimum by orthogonal projection.

The approximation problem allows us to vary the approximating space V, and this requires in general some non-linear optimization. The idea with this type of approximation is that V is the span of the output from a linear filter bank as in the KV model. Then one looks for a good approximation of $f(u)$ in terms of a polynomial in $\mathcal{P}_n(V)$, as exemplified in the KV model described above and in [14].

A2. Discrete Time Analysis of the H-N *(Wiener) System*

In the discrete time formalism above, the physical system f for the second-order H-N system corresponds to the time zero function

$$f(u) = \left(\sum_t g_t u_t\right) \qquad (19)$$

where (g_t) is a given square summable sequence. The system (19) is an homogenous *quadratic form* in the input u. It thus belongs to $\overline{\mathcal{P}}_2(H) \subset L^2(H)$.

To find the best finite dimensional approximation is trivial for the H-N system, since we can simply take V to be the one-dimensional space being spanned by $\xi(u) = \sum_t g_t u_t$, and then the physical system $f(u) = [\xi(u)]^2$ is actually an element of $\mathcal{P}_2(V)$. Therefore, the error in the approximation is automatically zero, and the only problem is to make any algorithm that identifies the right subspace V converge fast enough. These results agree well with the simulations above.

A3. Discrete Time Analysis of the N-H (Hammerstein) System

The general second-order N-H system corresponds to a function $f \in L^2$ given by

$$f(u) = \sum_t g_t (u_t)^2 \qquad (20)$$

where (g_t) is a given square summable sequence. The system (20) is a homogenous quadratic form in the input and $f(u)$ thus belongs to $\overline{\mathcal{P}}_2(H)$.

From the theory of quadratic forms, we know that we can always find an orthogonal basis such that the quadratic form in this basis takes a diagonal form. Hence, after a change of basis, the N-H system can be taken to represent *any* quadratic form in $\overline{\mathcal{P}}_2(H)$.

We can show that, in general, we require a high dimensional subspace V in order to properly approximate the N-H system (20). This “negative” result for the N-H system is less trivial than the “positive” result for the H-N system. It means that, under mild conditions on g_t in (20), we cannot expect to find a good approximation of such a general system in *any* $L^2(V)$ if the subspace V has a small dimension. Thus, in order to get good and small polynomial approximations for these kinds of systems, one should perhaps consider non-linear filter banks of subspaces of L^2 rather than just H.

To be more precise, let ε_M be the mean-square error for a best approximation of $f(u)$ in (20) by an element $\tilde{f}(u)$ in $L^2(V)$, where V is an M-dimensional subspace of H. That is, let

$$\varepsilon_M := \min_{\tilde{f},V} \mathbb{E}\left|\tilde{f} - f\right|^2, \qquad (21)$$

where V goes through all M-dimensional subspaces of H and $\tilde{f} \in L^2(V)$. Then we have that

$$\varepsilon_M \leq 2\sum_{t=1}^{\infty} g_t^2 - 2M^2 \left\|g\right\|_{\infty}^2, \qquad (22)$$

where $\left\|g\right\|_{\infty} = \max_t \left|g_t\right|$.

When $M = 1$, this bound (22) is moreover exact

$$\varepsilon_1 = 2\sum_{t=1}^{\infty} g_t^2 - 2\left\|g\right\|_2^{\infty} \qquad (23)$$

and this means that the optimal linear filter is a delay-filter of the form $y_t = u_{t-r}$ where r is the index such that g_r has a maximum absolute value. Thus, the best one-dimensional approximation corresponds to the space $V = \langle u_r \rangle$ spanned by the dominating basis vector u_r in the diagonalizing basis.

A3.1 Discussion of the Bounds (22) and (23)

For a "typical" g and a small dimension $M > 1$, the maximum reduction in error of $2M^2 g_r^2$ in (22), would be small compared to the total variance $Var(f) = 2\sum_t g_t^2$ of the input signal; we do not expect a bounded number of terms to dominate in an infinite sum. However, it can be the case that in the diagonalizing basis of $f(u)$, the sequence (g_t) has a small number of dominating terms. In this case, the finite dimensional approximation of $f(u)$ make sense. For instance, if the input signal u_t is band limited white noise and thus not orthogonal, then a small number of terms g_t^2 can effectively come to dominate in the $\sum_t g_t^2$ for the properly diagonalized version of f. This is illustrated in the last section.

A3.2 Proof of the Bounds

Without loss of generality, assume that the finite dimensional vector space V is spanned by a basis $\{\xi_1(u),\ldots,\xi_M(u)\}$ where

$$\xi_k(u) = \sum_{t=1}^{\infty} \xi_{k,t} u_t, \qquad (24)$$

which constitutes an orthonormal basis (ON-), i.e.,

$$\sum_t \xi_{i,t}\xi_{j,t} = \delta_{i,j}. \qquad (25)$$

Due to the "Wiener Chaos Decomposition" ([16]) of L^2, an ON-basis $\mathcal{B}_2$ for the closed subspace $\mathcal{P}_2(V)$ is obtained by taking products of Hermite polynomials with variables in $\mathcal{B}$. That is, by

$$\begin{aligned}\mathcal{B}_2 &= \{1\} \cup \{\xi_1, \xi_2, \ldots, \xi_M\} \cup \{\xi_1\xi_2, \xi_1\xi_3, \ldots, \xi_{M-1}\xi_M\} \\ &\cup \left\{\frac{1}{\sqrt{2}}\left(\xi_1^2 - 1\right), \frac{1}{\sqrt{2}}\left(\xi_2^2 - 1\right), \ldots, \frac{1}{\sqrt{2}}\left(\xi_M^2 - 1\right)\right\}.\end{aligned} \qquad (26)$$

We find the mean-square-best approximation of the physical system f by a function $\tilde{f} \in L^2(V)$ as the orthogonal projection $\tilde{f} = proj_{\mathcal{P}_2(V)}(f)$ of f onto $\mathcal{P}_n(V)$. Moreover, by the Pythagorean identity, $\varepsilon = \|f\|_2^2 - \|\tilde{f}\|_2^2$, we should choose V, or equivalently the basis $\mathcal{B}$, to maximize $R(V) = \|\tilde{f}\|_2^2$. By Parseval's identity, $R(V)$ equals the sum of squares of the coefficients when we express $\tilde{f}$ in the ON-basis $\mathcal{B}_2$. That is

$$R(V) = a^2 + \sum_{j=1}^{M} b_j^2 + \sum_{k<l} c_{kl}^2 + \sum_{m=1}^{M} d_m^2, \qquad (27)$$

where

$$\tilde{f} = a + \sum_{j=1}^{M} b_j \xi_j + \sum_{k<l} c_{kl}\xi_k\xi_l + \sum_{m=1}^{M} \frac{1}{\sqrt{2}} d_m \left(\xi_m^2 - 1\right). \qquad (28)$$

The coefficients a, b_j, c_{ij}, d_k are straightforward to compute if we recall that $u_t \sim N(0,1)$ i.i.d. and the orthogonality expressed in (25). The results are as follows:

$$\begin{aligned}
a &= \sum_{t=1}^{\infty} g_t \mathbb{E}\left(1 \cdot u_t^2\right) = \left(\sum_{t=1}^{\infty} g_t\right) \\
b_j &= \sum_{t=1}^{\infty} g_t \mathbb{E}\left(\xi_j \cdot u_t^2\right) = 0 \\
c_{kl} &= \sum_{t=1}^{\infty} g_t \left[\mathbb{E}\left(\xi_k\xi_l\right) \cdot u_t^2\right] = \\
&= \sum_{t=1}^{\infty} g_t \left[2\xi_{k,t}\xi_{l,t} + \left(\sum_{s=1}^{\infty} \xi_{k,s}\xi_{l,s}\right)\right] = \sum_{t=1}^{\infty} 2 g_t \xi_{k,t}\xi_{l,t} \\
d_m &= \sum_{t=1}^{\infty} g_t \frac{1}{\sqrt{2}}\left(2\xi_{m,t}^2 + \sum_s \xi_{m,s}^2 - 1\right) = \sqrt{2}\sum_{t=1}^{\infty} g_t \xi_{m,t}^2.
\end{aligned} \qquad (29)$$

Then we get an explicit expression for $R(V)$ as

$$\begin{aligned}R(V) = a^2 &+ 2\sum_{l=1}^{k}\left(\sum_t g_t \xi_{l,t}^2\right)^2 + \\ &+ 4\sum_{1\le k<l\le M}\left(\sum_t g_t \xi_{k,t}\xi_{l,t}\right)^2.\end{aligned} \qquad (30)$$

If $M = 1$ then the second term above is missing and since $\sum_t \xi_{i,t}^2 = 1$ the bound (23) follows; the maximum of $\left(\sum_t g_t \xi_{l,t}^2\right)^2$ is then clearly equal to g_r^2, where $g_r = \max_t |g_t|$.

For $M > 1$, the best bound is not immediate, but we note that the inequality of arithmetic and geometric means gives

$$\left(\sum_t g_t \xi_{i,t} \xi_{j,t}\right)^2 \leq \left[\sum_t |g_t| \cdot \frac{1}{2}\left(\xi_{i,t}^2 + \xi_{j,t}^2\right)\right]^2 \leq |g_r|^2 \quad (31)$$

where the last inequality has the same motivation as before, i.e. since $\sum_t \frac{1}{2}\left(\xi_{i,t}^2 + \xi_{j,t}^2\right) = 1$. This proves (22).

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